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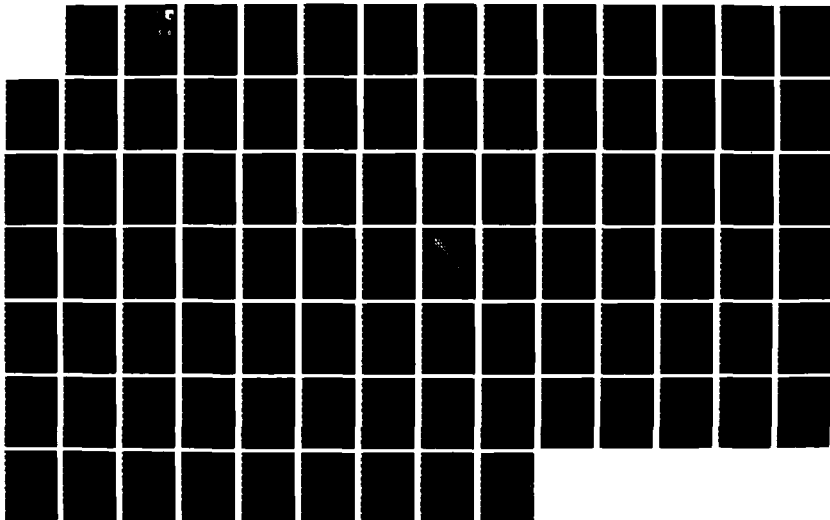
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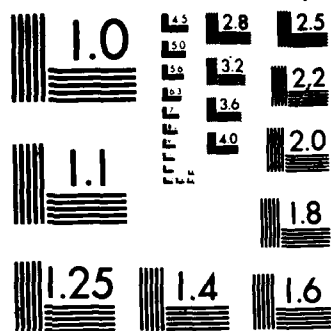
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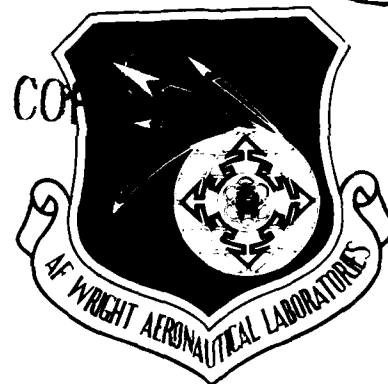




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SYMMETRICALLY LAMINATED BEAM AND PLATE
FINITE ELEMENTS WITH SHEAR DEFORMATION

Henry T.Y. Yang
Alex T. Chen

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April 1987

Final Report for Period June 1983 - October 1986

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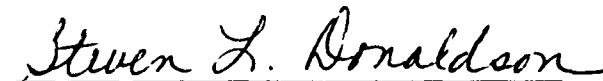
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<p>In this report, finite element formulations were developed for symmetrically laminated composite beams and plates for static and free vibration analyses.</p> <p>First, a formulation, an efficient solution procedure, a microcomputer program, and a graphics routine for a 12 d.o.f. anisotropic symmetrically laminated beam finite element including the effect of shear deformation is introduced. The emphasis of the formulation and solution procedure is for simplicity, efficiency, and easy implementation on microcomputers. The formulation, solution procedure, and the program have been evaluated by performing a systematic choice of examples; wherever possible, the present solutions are compared with alternative existing solutions.</p> <p>Second, an 18 d.o.f. triangular plate element in bending with anisotropic symmetrically laminated composite materials is formulated.</p> <p>To demonstrate and evaluate the present development, numerical computations on the static, free vibration, and buckling analyses of a series of anisotropic symmetrically laminated composite materials is formulated.</p> <p style="text-align: right;">(continues)</p>				
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laminated plate problems have been performed using a microcomputer. Further testing and evaluation of the previously formulated beam element for more cases were conducted and the results were compared with those obtained using the present plate elements. The comparison indicated that both types of element formulations are sufficiently simple and accurate, and that the numerical methods are efficient for microcomputers.

Finally, an approach was presented in the formulation of a 36 d.o.f. symmetrically laminated composite triangular plate element including the effect of shear deformation for free vibration analysis. The strain energy of plates can be expressed as the sum of the flexural and shear strain energies. The total displacement is expressed as the sum of the displacement due to bending and that due to shear deformation. Thus only the displacement due to bending appear in the flexural strain energy and the displacement due to shear deformation appear in the shear strain energy. Numerical results for natural frequencies for a range of different isotropic, orthotropic and anisotropic plates with various thickness-to-length ratios were obtained and compared with various alternative solutions available in the literature to demonstrate the valid range of applicability of this approach for the free vibration of plates with the effect of shear deformation.

11. Title:

Symmetrically Laminated Beam and Plate Finite Elements with Shear Deformation

FOREWORD

This report was prepared by T.Y. Yang and Alex T. Chen, School of Aeronautics and Astronautics, Purdue University for the Mechanics and Surface Interactions Branch (AFWAL/MLBM), Nonmetallic Materials Division, Materials Laboratory, Air Force Wright Aeronautical Laboratories, Wright-Patterson AFB, Ohio. The Work was performed under Contract Number F33615-83-C-5076, Project Number 2419, Task 01, Work Unit 57.

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LIST OF SYMBOLS

B_1, B_2	first and second bending modes
b	width of the beam
D_{ij}	bending stiffnesses
DS_{44}	transverse shear stiffness
d_{11}, d_{16}	compliances
E	modulus of elasticity
F	force
F_i	nodal forces (point loads and moments, etc.)
F_1, F_2	point loads at the respective beam element nodes
f_1, f_2, f_3, f_4	shape functions
Hz	Hertz
h	height, depth, or thickness of the beam or laminate
I	moment of inertia about the centroid
J	polar mass moment of inertia about the centroidal axis
k	stiffness
L	length of the beam
ℓ	length of the beam element
M_x, M_y, M_{xy}	plate bending moments
M_1, M_2	bending moments at the respective beam element nodes
m	mass per unit length of the beam element
P	applied point load
Q_x	shear force
q	nodal degrees of freedom
s	symmetric
T	kinetic energy
T_1	first torsional mode
T_1, T_2	torques at the respective beam element nodes 1 and 2

T_1, T_2	rate of torque with respect to beam axis at the respective beam element nodes 1 and 2
t	time
t_i	ply thickness at the i th layer
U	potential energy
W	transverse deflection
W_b	beam transverse deflection due to bending deformation
W_{b1}, W_{b2}	transverse deflection due to bending deformation at the respective beam element nodes 1 and 2
W_s	beam transverse deflection due to shear deformation
W_{s1}, W_{s2}	transverse deflection due to shear deformation at the respective beam element nodes 1 and 2
x	x-coordinate (along the beam axis)
y	y-coordinate (along the beam width)
z	z-coordinate (along the beam thickness or depth)
α	shear coefficient
γ_x	shear angle
$\epsilon_x, \epsilon_y, \epsilon_{xy}, \epsilon_{xz}$	inplane and transverse strains
θ	ply angle
θ_{b1}, θ_{b2}	bending slope due to bending deformation at the respective beam element nodes 1 and 2
θ_{s1}, θ_{s2}	bending slope due to shear deformation at the respective beam element nodes 1 and 2
π	pi
ρ	mass per unit volume
$\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{xz}$	inplane and transverse stresses
τ_1, τ_2	twisting angle at the respective beam element nodes 1 and 2
τ_1, τ_2	rate of twist with respect to beam axis at the respective beam element nodes 1 and 2
ϕ	twisting angle
ψ	shear angle
ω	harmonic frequency

SECTION I

INTRODUCTION

During the last three decades, the research and development in finite element method have grown from infancy to maturity and this method has revolutionized the methods of structural analysis and design. Such development has been well documented in the texts by, for example, Martin [1], Zienkiewicz [2], and Gallagher [3].

During the last decade, the development of calculators and desktop microcomputers has grown at a very rapid rate and it becomes an obvious trend that more simple daily structural designs will be performed using microcomputers. To cope with such trend, works have currently been underway to convert some general purpose finite element programs suitable for microcomputers.

During the last two decades, the research and development of laminated composite structures has grown at an extremely rapid pace and it becomes an obvious trend that more and more composite material will be used in the design of structures when the weight and strength are of primary consideration. The fundamental development in the mechanics of composite materials has been documented by, for example, Tsai [4] and Jones [5]. The basic theory of the mechanics of composite materials, particularly for laminated plates, has been widely used in finite element formulations. Thus, it is common that existing isotropic and homogeneous finite elements also have the capability of treating

laminated composite materials. A survey of recent research in the analysis of composite plates including finite element methods can be found, for example, in [6] by Reddy. An evaluation of finite element software for stress analysis of laminated composites was given in Reference [7].

In view of the trend of growing use of microcomputers in the common engineering office, it appears that there is a definite need for research and development work to tailor and simplify the basic formulations for laminated composite finite elements into a form suitable for programming using microcomputers. As a first step to respond to this need, in Chapter 2, a simplified symmetrically laminated beam-type finite element is developed and programmed for a microcomputer. The element is assumed to have six degrees of freedom at each of the two ends: transverse deflection and slope due to bending and shear, respectively; and a twisting angle and its derivative with respect to beam axis.

This program implemented in a microcomputer is capable of performing stress analysis of symmetrically laminated beam structures with a single or combined effect of bending moment, twisting moment, and shear deformation, and with arbitrary loading and boundary conditions. It is also capable of performing free vibration analysis without shear deformation. For static analysis, the program has the capability of providing both numerical data and graphical plots of the distributions of displacements, bending and twisting moments, ply stresses, and the portions contributed by shear deformation. For free vibration analysis, the program gives the natural frequencies and mode shapes.

The simple homogeneous and isotropic beam finite element formulation is essentially the same as that traditionally formulated by the slope-deflection equations for beams. Such an element was extended by, among others, McCalley [8], Archer [9], and Kapur [10], to include the effects of shear deformation. Archer's element has two degrees of freedom at each of the two nodes: total transverse deflection due to combined effect of bending and shearing deformations and its derivative with respect to beam axis. On the other hand, Kapur developed a beam element where the bending and shearing deformations were considered separately with separate associated degrees of freedom. Thus the element has four degrees of freedom at each of the two nodes: transverse deflection and its derivative with respect to the beam axis due to bending and shearing deformations, respectively. For the present laminated composite beam finite element, Kapur's type of approach was used to account for the effect of shear deformation. The homogeneous anisotropic beam theory which considered the coupling between bending and torsion was given by Lekhnitskii [11]. Such coupling was incorporated in the present formulation. A beam finite element formulation for laminated composite material with a single fiber orientation and shear deformation was given by Teh and Huang [12] for free vibration analysis. In that element, six degrees of freedom were assumed at each node: total deflection, total slope, twist derivative of bending slope with respect to beam axis, second derivative of bending slope with respect to beam axis, angular displacement and angle of twist. This paper differs from Reference [12] in that different types of degrees of freedom are assumed aiming at performing static and free vibration

analyses using a microcomputer in a simpler, more efficient and general fashion.

In Chapter 3, a simple yet efficient formulation for a laminated composite plate finite element was developed and pertinent efficient numerical algorithms were adopted so that the development is suitable for use in stand alone desktop microcomputers for structural analysis and design. Furthermore, a previously formulated 8 d.o.f. beam finite element [13] (see Figures 1.1 and 1.2) were further evaluated and compared with the more sophisticated plate finite element.

In the finite element formulations, an 18 d.o.f. triangular plate element in bending with anisotropic symmetrically laminated composite materials was formulated. The triangular plate finite element described in Figure 1.3 has 6 d.o.f.'s at each of the 3 nodes: transverse deflection, its first derivatives with respect to ξ and η , respectively, its second derivatives with respect to ξ and η , respectively, and its twist derivative with respect to ξ and η , where ξ and η are the local coordinates. This element was developed by Cowper et al. [14] for isotropic materials. It has now been extended to include the effect of laminated composites for use on the microcomputer. This element is highly efficient as it meets the convergence criteria [15] and the normal slope varies cubically along the interelement boundaries with full compatibility. Furthermore, the integrations of the stiffness coefficients can be expressed in a closed form. Both features made this element well-suited for microcomputers.

In the development of numerical procedures, emphasis has been placed upon the minimization and condensation of memory storage and

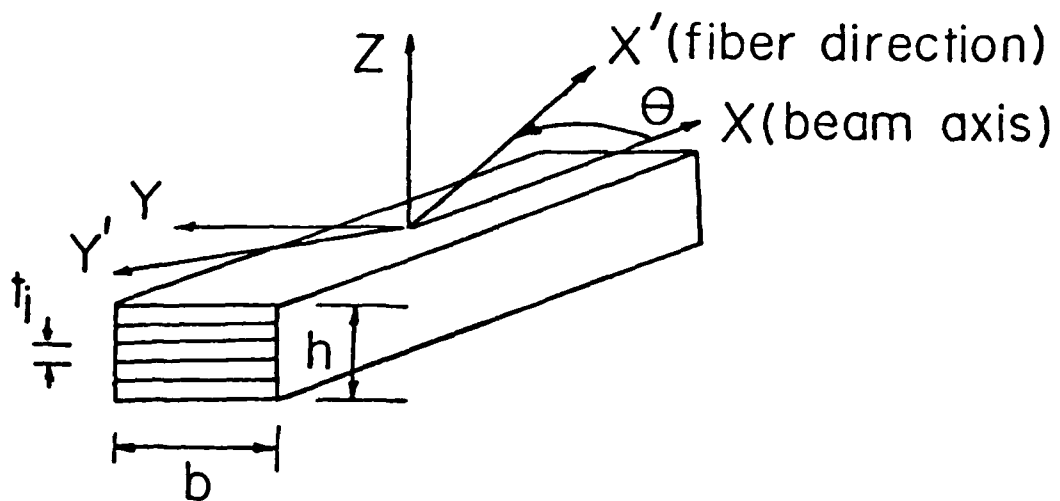


Figure 1.1 Positive ply orientation.

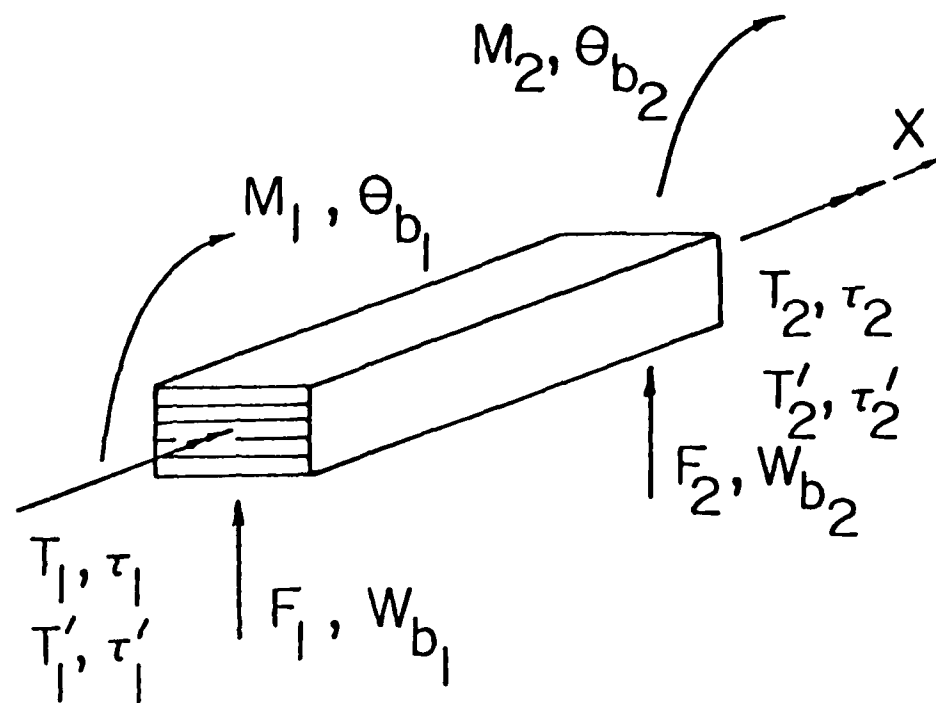


Figure 1.2 8 d.o.f. beam finite element.

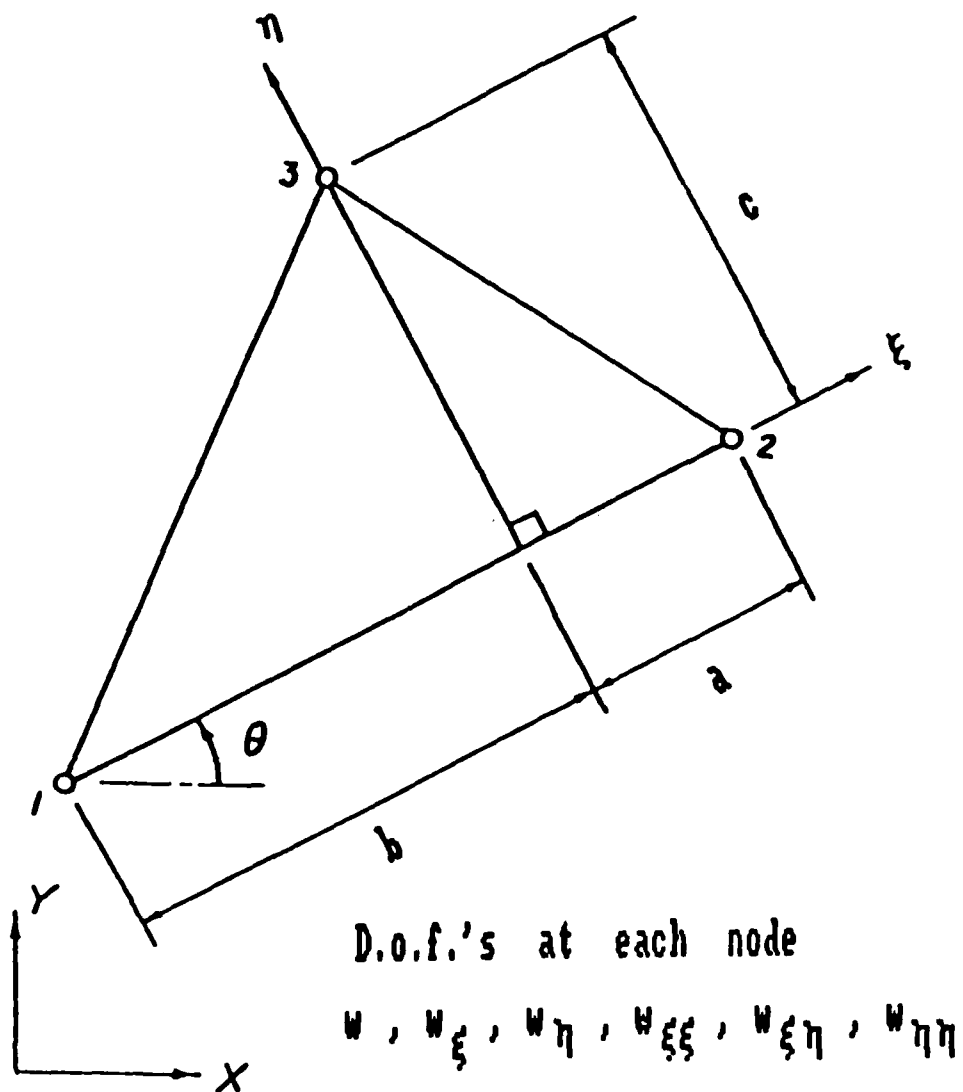


Figure 1.3 18 d.o.f. plate finite element.

efficiency of computation. The minimization of memory storage is achieved by the use of banded and profile equation solvers for the static case. For the eigenvalue problem, a symmetrical matrix eigenvalue solver is used to minimize memory storage. For the special case of lumped mass matrices, a condensation technique eliminating the zero terms along the diagonal of the mass matrix is used to reduce the total number of d.o.f.'s or equations. For efficiency of computation, only a few of the element stiffness matrices need be calculated provided the structure has repeating elements. Certain 20x20 transformation matrices can be inverted a priori without the use of numerical routines provided the elements are right triangles.

To demonstrate and evaluate the present formulations and numerical techniques, a series of static, free vibration, and buckling analyses of anisotropic symmetrically laminated plate problems have been performed using a microcomputer. To further test the previously formulated beam finite element [13], results of the beam element has been obtained and compared to those of the plate element. As will be demonstrated in Chapter 3, the present development is simple, efficient, and practical.

It has been known that the effect of shear deformation must be included in the finite element formulation. There are a number of ways to include the effects of shear deformation. A popular approach is to use revised plate theory by Reissner [16]. However, in some formulations the problem known as "shear locking" is encountered in the inclusion of the effect of transverse shear deformation. Some techniques such as reduced integration, penalty methods, energy balancing,

mixed formulations and others have been proposed to alleviate the shear locking.

In Chapter 4, an approach was presented in the formulation of a 36 d.o.f. symmetrically laminated composite triangular plate element which does not present the shear locking problem. This is an extended version of the 18 d.o.f. isotropic triangular thin plate element by Cowper, Kosko, Lindberg, and Olson [14]. This formulation is also a generalization of the approach by Kapur [10] for an isotropic beam element with the effect of shear deformation. In this approach, the effect of shear deformation was incorporated into the formulations by expressing the strain energy of deformation as the sum of the thin plate flexural energy and the energy of transverse shear deformation, and also by expressing the total transverse displacement as the sum of the displacements due to bending alone and that due to shear deformation alone. Furthermore, only the displacement due to bending alone appears in the flexural strain energy and the displacement due to shear deformation appears in the shear strain energy. A similar approach for the inclusion of transverse shear deformation has been presented by Bhashyam and Gallagher [17] for the static analysis of isotropic plates using an 18 d.o.f. triangular finite element. In their approach the strain energy expression is the sum of the bending and shear deformation energies. The strain energy expression is in terms of the total displacement and that due to bending alone. Only the displacement due to bending alone appear in the flexural strain energy. However, the displacement due to shear deformation in the shear strain energy is expressed as the difference of the total displacement and the displacement due to bending.

To evaluate the range of applicability and efficiency of the present formulation, a variety of isotropic, orthotropic and anisotropic plates with various thickness-to-length ratios were obtained and compared with various alternative solutions available in the literature to demonstrate this approach for the free vibration of plates with the effect of shear deformation and rotatory inertia.

SECTION II

STATIC AND DYNAMIC FORMULATION OF A SYMMETRICALLY LAMINATED BEAM FINITE ELEMENT FOR A MICROCOMPUTER

1. Symmetrically Laminated Finite Element Formulation

In this formulation, a symmetrically laminated composite beam was considered. The beam is made of layers of orthotropic material in which the orthotropic axes of layer may be oriented at an arbitrary angle with respect to the beam axis and in which the layers are stacked symmetrically with respect to the midsurface of the beam (see Figure 1.1). For the case of a symmetrically laminated composite beam, there exists coupling between the bending and twisting curvatures.

1.1. Formulation

The stress-strain relation due to bending deformation only can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix} \quad (2.1)$$

where the σ 's are the stress components, ϵ 's are the strain components, and the Q_{ij} 's are the inplane ply stiffnesses.

The stress-strain relation due to transverse shear deformation is given by

$$\sigma_{xz} = Q_{44} \epsilon_{xz} \quad (2.2)$$

For the general laminate, the linear strain distribution across the thickness of the laminate is given by

$$\begin{aligned} e_x &= \epsilon_x^0 + \epsilon_x \\ e_y &= \epsilon_y^0 + \epsilon_y \\ e_{xy} &= \epsilon_{xy}^0 + \epsilon_{xy} \end{aligned} \quad (2.3)$$

where ϵ^0 's are the strains due to inplane forces, and ϵ 's the strains due to bending deformation. The moment-curvature and inplane stress-strain relations for the general laminate is given by

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \epsilon_{xy}^0 \\ k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad (2.4)$$

where N's are the inplane forces, M's the bending and twisting moments,

k 's the bending and twisting curvatures, A_{ij} 's the inplane moduli, B_{ij} 's the coupling moduli, and D_{ij} 's the flexural moduli.

For the case of a symmetrically laminated plate, the coupling moduli B_{ij} vanish. Thus the flexural and membrane behaviors become uncoupled.

For the case of the beam, no bending moment M_y exists. It is noted, however, that the curvature k_y is assumed to be nonzero. Thus, the moment curvature relation may be rewritten as

$$\begin{Bmatrix} M_x \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} - \frac{D_{12}^2}{D_{22}} & D_{16} - \frac{D_{12}D_{26}}{D_{22}} \\ D_{16} - \frac{D_{12}D_{26}}{D_{22}} & D_{66} - \frac{D_{26}^2}{D_{22}} \end{bmatrix} \begin{Bmatrix} k_x \\ k_{xy} \end{Bmatrix} \quad (2.5)$$

The transverse shear force-strain relation is given by

$$Q_x = \alpha D_{44} \gamma_x \quad (2.6)$$

where Q_x is the transverse shear force, α the shear coefficient, D_{44} the transverse shear stiffness and γ_x the transverse shear strain.

The strain energy expression is given as

$$U = \frac{b}{2} \int \{M_x k_x + M_{xy} k_{xy}\} dA + \frac{b}{2} \int \{Q_x \gamma_x\} dA \quad (2.7)$$

where $k_x = \frac{d^2 w_b}{dx^2}$, $k_{xy} = 2 d\phi/dx$,

$\gamma_x = dw_s/dx$, b = width of beam, A = area

w_b = displacement due to bending deformation,

w_s = displacement due to shear deformation,

ϕ = twist.

The kinetic energy expression for a plate can be written in the general terms as

$$T = \frac{\rho}{2} \iiint \dot{w}^2 (x,y,t) dx dy dz \quad (2.8)$$

where ρ is the mass density per unit volume. The dot represents derivative with respect to time.

For the case of the beam in the absence of shear deformation, the deflection function may be written as

$$W(x,y) = w_b(x) + y \phi(x) \quad (2.9)$$

Substituting Equation (2.9) into Equation (2.8) and performing integration result in,

$$T = \frac{\rho A}{2} \int_0^{\ell} \dot{w}_b^2 (x) dx + \frac{J}{2} \dot{\phi}^2 (x) dx \quad (2.10)$$

where A = cross-sectional area of the beam, and J = polar mass moment of inertia about the centroidal axis.

Substituting the kinetic energy expression Equation (2.10) and the strain energy expression Equation (2.7) into the Lagrange's equation

$$F_i = \frac{dt}{dt} \frac{dT}{d\dot{q}_i} + \frac{dU}{dq_i} \quad (2.11)$$

yields

$$\{F\} = [k]\{q\} + [m]\{\ddot{q}\} \quad (2.12)$$

where $[k]$ is the stiffness matrix; $[m]$ is the consistent mass matrix, (which can also be derived in other forms such as "lumped"); and $\{q\}$ is the vector for the six nodal degrees of freedom.

Assuming free vibration with simple harmonic frequency ω , Equation (2.12) becomes

$$\{[k] - \omega^2[m]\} \{q\} = 0 \quad (2.13)$$

The consistent mass matrix and the stiffness matrix including shear deformation are given by

$$[M] = \begin{bmatrix} M11 & M12 & & & M15 & M16 & & \\ & M22 & & & M25 & M26 & & \\ & & M33 & M34 & & & M37 & M38 \\ & & & M44 & & & M47 & M48 \\ & & & & M55 & M56 & & \\ & & & & & M66 & & \\ & & & & & & M77 & M78 \\ & & & & & & & M88 \end{bmatrix} \quad (2.14)$$

where

$M11 = 156 F$; $M12 = -22L F$; $M15 = 54 F$; $M16 = 13L F$; $M22 = 4L^2 F$;
 $M25 = -13L F$; $M26 = -3L^3 F$; $M33 = 156 G$; $M34 = -22L G$; $M37 = 54 G$;
 $M38 = 13L G$; $M55 = 156 F$; $M56 = 22L D$; $M66 = 4L^2 F$; $M77 = 156 G$;
 $M78 = 22L G$; $M88 = 4L^2 G$; $G = JL/420$; $F = mL/420$;
 J = mass polar moment of inertia per unit length of the element;
 m = mass per unit length of the element, and

$$[k] = \begin{bmatrix}
 K11 & K12 & & & K16 & K17 & K18 & & & & K1C \\
 & K22 & & & K25 & K26 & K27 & K28 & & K2B & K2C \\
 & & K33 & K34 & & & & & K39 & K3A & \\
 & & K44 & & & & & & K49 & K4A & \\
 & & & & K55 & K56 & & K58 & & K5B & K5C \\
 & & & & & K66 & K67 & K68 & & K6B & K6C \\
 & & & & & & K77 & K78 & & & K7C \\
 & & & & & & & K88 & & K8B & K8C \\
 & & & & & & & & K99 & K9A & \\
 & & & & & & & & & KAA & \\
 & & & & & & & & & & KBB & KBC \\
 & & & & & & & & & & & KCC
 \end{bmatrix} \quad (2.15)$$

where

$K11 = -K17 = K77 = 12 D11 \cdot L^3$; $K12 = K18 = -K27 = -K78 = -6 D11 \cdot L^2$;
 $K22 = K88 = 4 D11 \cdot L$; $K28 = 2 D11 \cdot L$; $K16 = -K1C = -K25 = K2B = K58 =$
 $-K67 = K7C = -K8B = 6 D66 \cdot L/5$; $K56 = K5C = -KBC = -D66 \cdot L/10$; $K66 = KCC =$
 $2 D66 \cdot L/15$; $K6C = D66 \cdot L/30$; $K33 = -K39 = K99 = 6 S/5L$; $K34 = K3A =$
 $-K9A = -K49 = -A/10$; $K44 = KAA = 2L S/15$; $K4A = -L S/20$; $S = \alpha D44$;

$$D11^* = b (D11-D12^2/D22); D16^* = b(D16-D12 D26/D22); D66^* = b(D66-D26^2/D22).$$

1.2 Description of the Element

The present symmetrically laminated beam element is described in Figure 2.1. The element possesses 6 degrees of freedom at each of the two nodes: the deflection due to bending W_b , the deflection due to shear deformation W_s , and their respective derivatives with respect to the x-axis $\theta_b (= -dW_b/dx)$ and $\theta_s (= -dW_s/dx)$, and the twisting angle $\phi (= \tau)$ and its derivative $\tau' (= d\phi/dx)$. The displacement functions for W_b , W_s and ϕ are assumed as,

$$\begin{aligned} W_b(x) &= f_1(x)W_{b1} + f_2(x)\theta_{b1} + f_3(x)W_{b2} + f_4(x)\theta_{b2} \\ W_s(x) &= F_1(x)W_{s1} + f_2(x)\theta_{s1} + f_3(x)W_{s2} + f_4(x)\theta_{s2} \\ \phi(x) &= f_1(x)\tau_1 + f_2(x)\tau_1' + f_3(x)\tau_2 + f_4(x)\tau_2' \end{aligned} \quad (2.16)$$

where the shape functions are in terms of

$$\begin{aligned} f_1(x) &= 1+2(x/\ell)^3-3(x/\ell)^2, \\ f_2(x) &= -x+2x^2/\ell-x^3/\ell^2, \\ f_3(x) &= 3(x/\ell)^2-2(x/\ell)^3, \\ f_4(x) &= -x^3/\ell^2+x^2/\ell. \end{aligned} \quad (2.17)$$

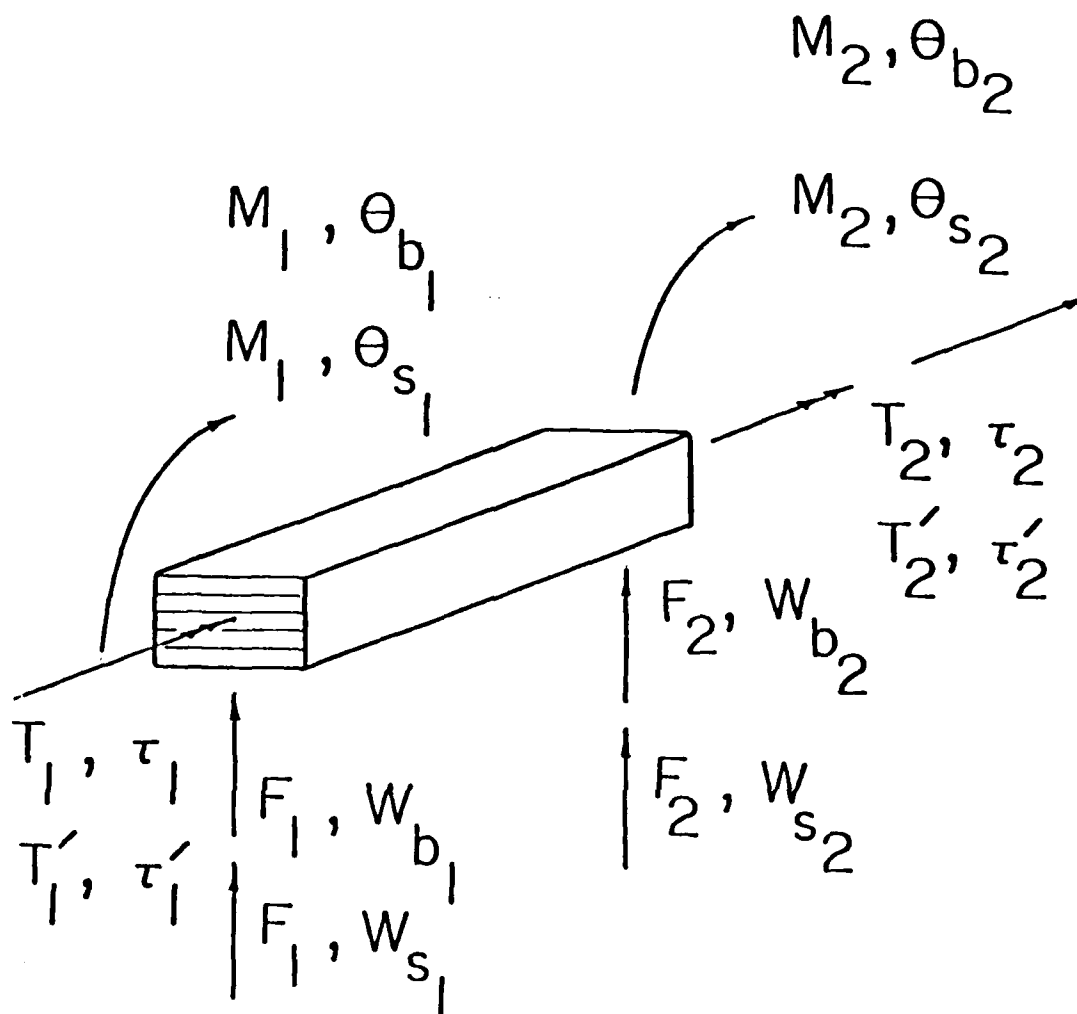


Figure 2.1 The 12 d.o.f. element with the effect of shear deformation.

2. Microcomputer Program

The formulation of the present 12 d.o.f. symmetrically laminated composite beam finite element has been coded into a microcomputer in Basic language. For static analysis, the program computes the transverse deflections and its slope distribution due to bending and shearing deformation along the beam axis. It then computes the moment, shear force, and torque distributions along the beam axis. It finally computes the inplane normal and shearing stresses, and transverse shearing stress distributions through the laminate thickness. For free vibration analysis, the program computes the natural frequencies and the corresponding mode shapes along the beam axis. A flowchart of the program is given in Figure 2.2.

For the static case, the program uses a symmetrically banded matrix solver which reduces both computing time and memory storage. The program also has a graphics routine which plots the distributions of the various aforementioned quantities on the microcomputer monitor screen. There are several devices which can be used to accelerate the computational speed. The first method is to compile the program using the available compilers on the market. The second method is to use a machine-dependent arithmetic chip which accelerates to store the results in a file, which can be printed later. Similarly, the plots may be stored in a file and recalled, and a hard copy of the plot can be obtained using available "screen dumps."

For the free vibration, the program uses a Jacobi eigenproblem solver. It computes the natural frequencies and plots the mode shapes.

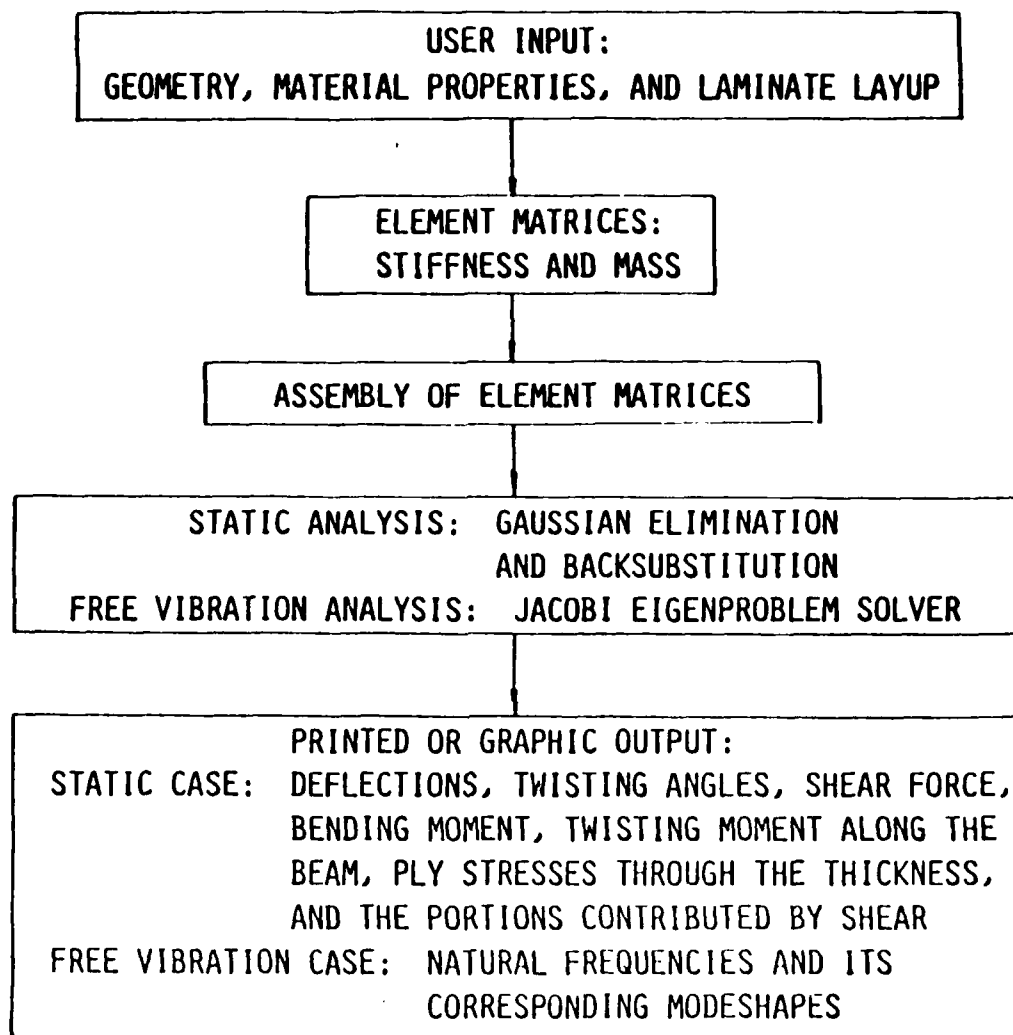


Figure 2.2 Flow chart for the microcomputer program.

This program has been written for the IBM PC. The minimum configuration needed to run the program requires a 128K memory storage, a single-sided disk drive, and a monitor. Additional hardware and software can enhance the performance. The compiler provided by IBM is the IBM Basic Compiler Version 1.00. The additional hardware is the 8087 arithmetic chip which is used in conjunction with the compiler. The "screen dump" software is provided by the DOS System Diskette.

3. Evaluative Analysis

The formulation, solution procedure, and program have been evaluated by performing the following examples with the alternative solutions for comparison.

3.1. Static Analysis

3.1.1. Homogeneous, Isotropic Cantilever Beam Under End Load P

To test the portion of the formulation which accounts for the effect of shear deformation, an example of a homogeneous, isotropic cantilever beam under end load P was first analyzed. The results obtained using one 12 d.o.f. element for the end deflection due to the effect of shear deformation only are given in Table 3.1 for two different values of shear coefficient α , as defined in Reference [18]. For $\alpha = 0.667$, an alternative exact solution available in the text by Timoshenko [19] and a one-element solution by Archer [9] are shown in the table. The four values for the end deflection by the four different methods are seen to be in excellent agreement. For $\alpha = 0.867$, a one-element by Archer [13] and a solution based on the energy method by Popov [20] are also shown in the table; the present solution is 4%

Table 2.1 Maximum deflection W_s in $Plh^2/4EI$ due to shear deformation only for a homogeneous isotropic cantilever beam under end load.

	Shear Coefficient α (defined in Ref. 18)	
	$\alpha=0.667$	$\alpha=0.867$
Exact Solution (14)	1.33	---
One 12 D.O.F. Element	1.33	1.0
Archer (9)	1.34	1.17
Based on Strain Energy (15)	---	1.04

lower than that by Popov and 17% lower than that by Archer. It is noted that the difference between the present solution and that by Archer may be due to the difference in the boundary conditions at the fixed end. In the former, it is assumed that $dW_s/dx \neq 0$, and in the latter, it is assumed that $dW/dx + \psi = 0$, where W_s , is the deflection due to shear deformation; W is the total deflection; and ψ is the shear angle.

3.1.2. Simply-Supported Rectangular 3 Layer (0/90/0) Cross-Ply Laminated Plate With Infinite Aspect Ratio (Wide Beam) With Shear Deformation

The second example chosen was a simply supported rectangular 3 layer (0/90/0) cross ply laminated plate with infinite aspect ratio (wide beam) subjected to a sinusoidally distributed load,

$$P = P_0 \sin \frac{\pi x}{\ell} \quad (2.18)$$

where ℓ is the length and x is the coordinate along the span. Due to symmetry, only half of the beam need be modeled; and 1,2,3,6, and 8 elements were used, respectively. The work equivalent loads based on the integration of the products of the shape functions and the distributed load were used. The central deflection due to the effect of shear deformation only was obtained and given in Table 2.2

An alternative analytical solution provided by Pagano and Whitney [21] is also shown in Table 2.2. It is seen that the present solution practically converged to that given in Reference [21] at the 6 element level.

Table 2.2 Non-dimensionalized maximum deflection due to shear deformation only of a simply-supported rectangular 3 layer (0/90/0) cross-ply laminated plate with infinite aspect ratio (wide beam) under a sinusoidal load $P = P_0 \sin \frac{\pi x}{\ell}$.

Number of Elements for half of beam	Non-dimensionalized maximum deflection $\frac{w_s \alpha D_{44}}{P_0 \ell^2}$	Pagano and Whitney (21) $\frac{w_s D_{44}}{P_0 \ell^2}$
1	0.3599	0.4053
2	0.4463	
4	0.4014	
6	0.4057	
8	0.4056	

3.1.3. Anisotropic 16 Layer (45₄/-45₄)s Laminated Cantilever Beam Under End Load P

The third example chosen was an anisotropic 16 layer laminated cantilever beam under an end load P without the effect of shear deformation. The results for the distributions of twisting angle and deflection due to bending, respectively, along the beam length were obtained using one element only and were plotted in Figure 2.3. With the use of the displacement functions in Equation (1), the displacement can be interpolated anywhere within the element. The twisting angle and the deflection were nondimensionalized as $(4\tau/Pl^2d_{16})$ and $(3W/Pl^3d_{11})$, respectively, where τ = twisting angle; l = length of the beam, d_{11} , d_{16} are flexural compliances; and W = deflection.

An alternative analytical solution for this problem treating the beam as homogeneous and anisotropic by Lekhnitskii [11] is also plotted in Figure 2.4 for comparison. The agreement is good.

The normalized inplane ply stress (σ_{xx} , σ_{xy} , and σ_{yy}) distributions through the laminate thickness were plotted in Figure 2.4. It is noted that force equilibrium is satisfied, and the bending moment due to end load P is recovered from summing the moments due to σ_{xx} . It is also noted that the summation of the moments due to σ_{xy} and σ_{yy} , respectively, give zero resultant moments.

3.1.4. Quasi-Isotropic (0/90/+45)s Laminated Beam Under Four Point Bending

The fourth example chosen was a quasi-isotropic (0/90/+45)s laminated beam under four point bending. The results for the distribution of the transverse shear stress through the laminate thickness were

NON-DIMENSIONAL BEAM DISPLACEMENT

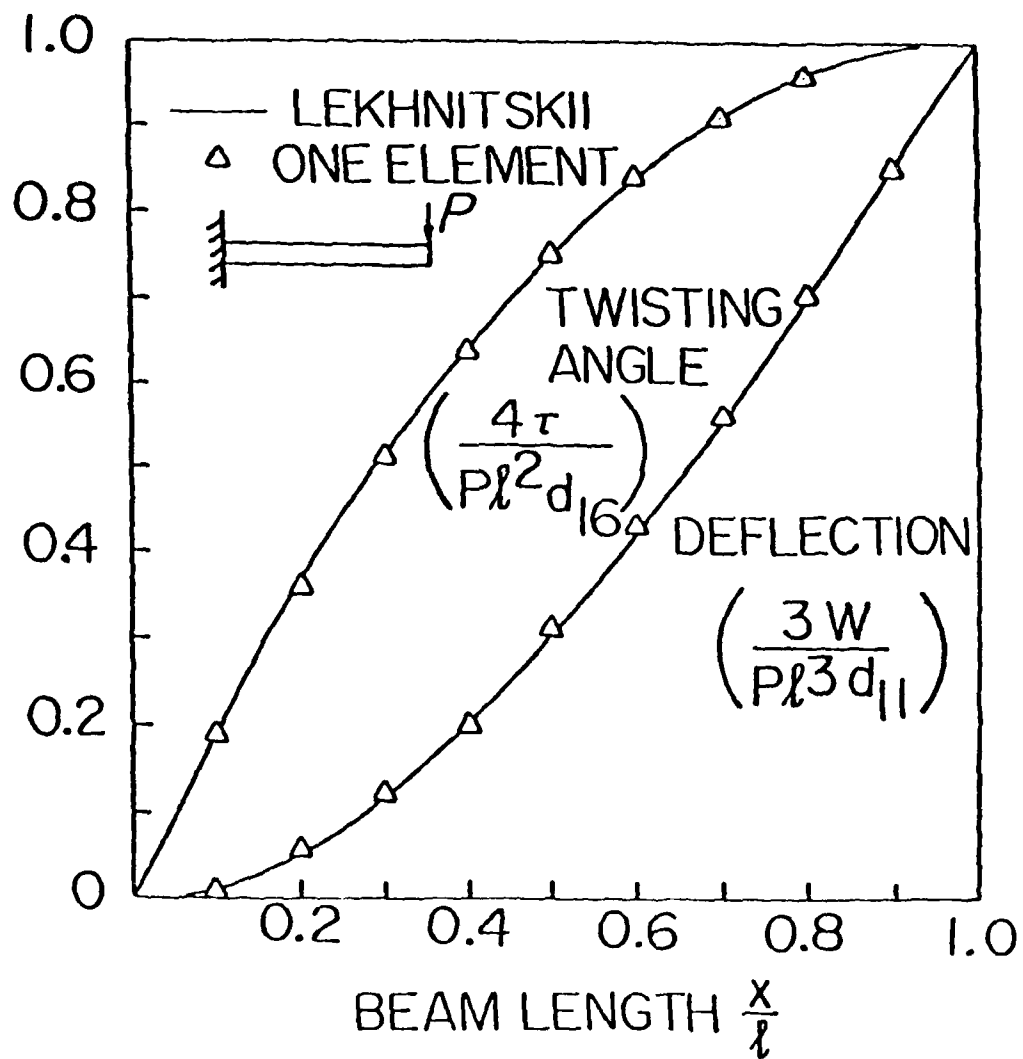
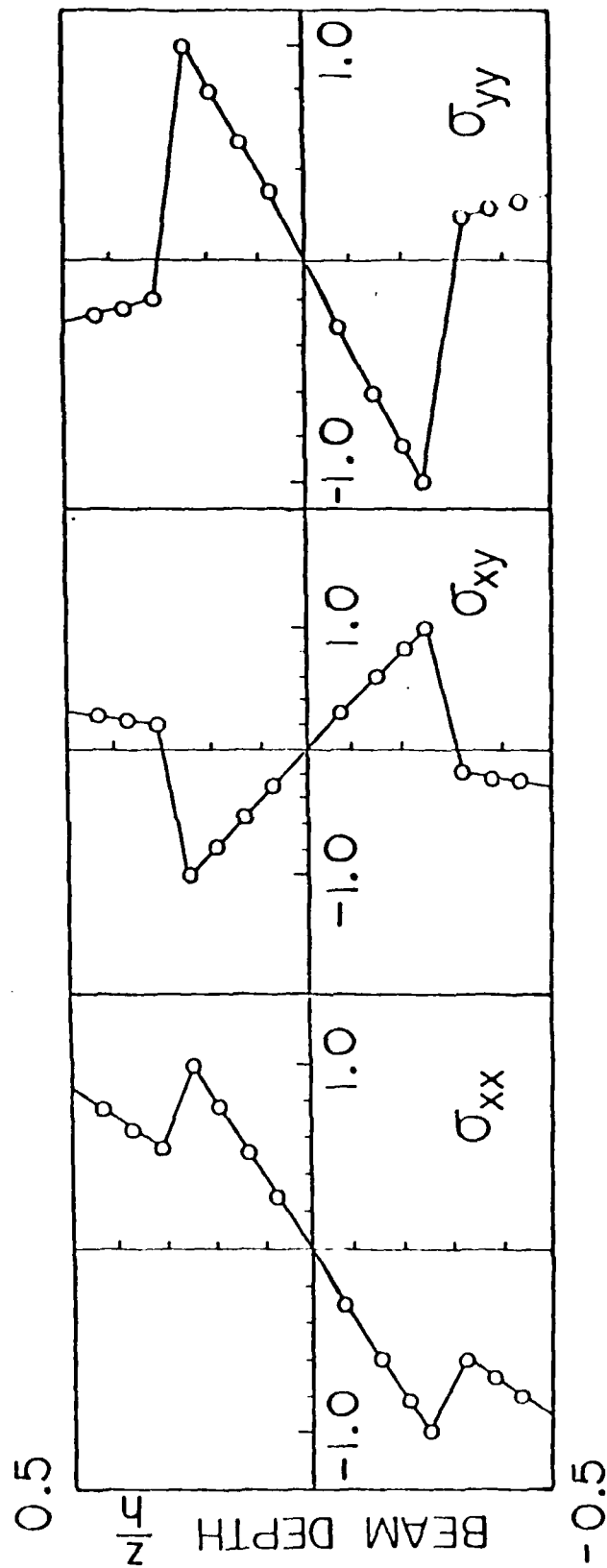


Figure 2.3 Distribution of deflection and twisting angle due to bending for a 16 ply $(45_4/-45_4)_s$ anisotropic laminated cantilever beam under end load P .



NORMALIZED STRESSES

Figure 2.4 Distribution of inplane ply stresses through the thickness for a 16-ply $(45_4/-45_4)_s$ anisotropic laminated cantilever beam under end load P .

obtained using two elements, and were plotted in Figure 2.5. The shear stress has been non-dimensionalized as $(\sigma_{xz}/Q_x h^2)$. The resultant shear force calculated from under the shear stress curve is 1.03, 3% higher than unity. The result given by Whitney [22] is also shown in the figure.

3.1.5. Anisotropic 16 Layer $(45_4/-45_4)_s$ Laminated Cantilever Beam Under End Load P With Shear Deformation

The final example chosen was the same as that given in (2.3.1.3) but with the effect of shear deformation. The results for the distributions of twisting angle and deflection due to bending as well as shear deformation along the length of the beam were obtained using one element and were plotted in Figure 2.6. Again, the twisting angle and the deflections were nondimensionalized the same way as were those done in Figure 2.3. The ply stresses are the same as those in Figure 2.4

3.2. Free Vibration Analysis

3.2.1. Homogeneous, Isotropic, Thin Cantilever Plate Without Shear Deformation

To test the portion of the formulation which accounts for the special case of the homogeneous, isotropic materials, an example of an aluminum cantilever plate without shear deformation was first analyzed. The results obtained using four present elements are given in Table 2.3. The experimental results used for comparison are given by Crawley [23]. The present shows good agreement for the bending frequencies is due to the present modeling of the two-dimensional plate as a one-dimensional beam.

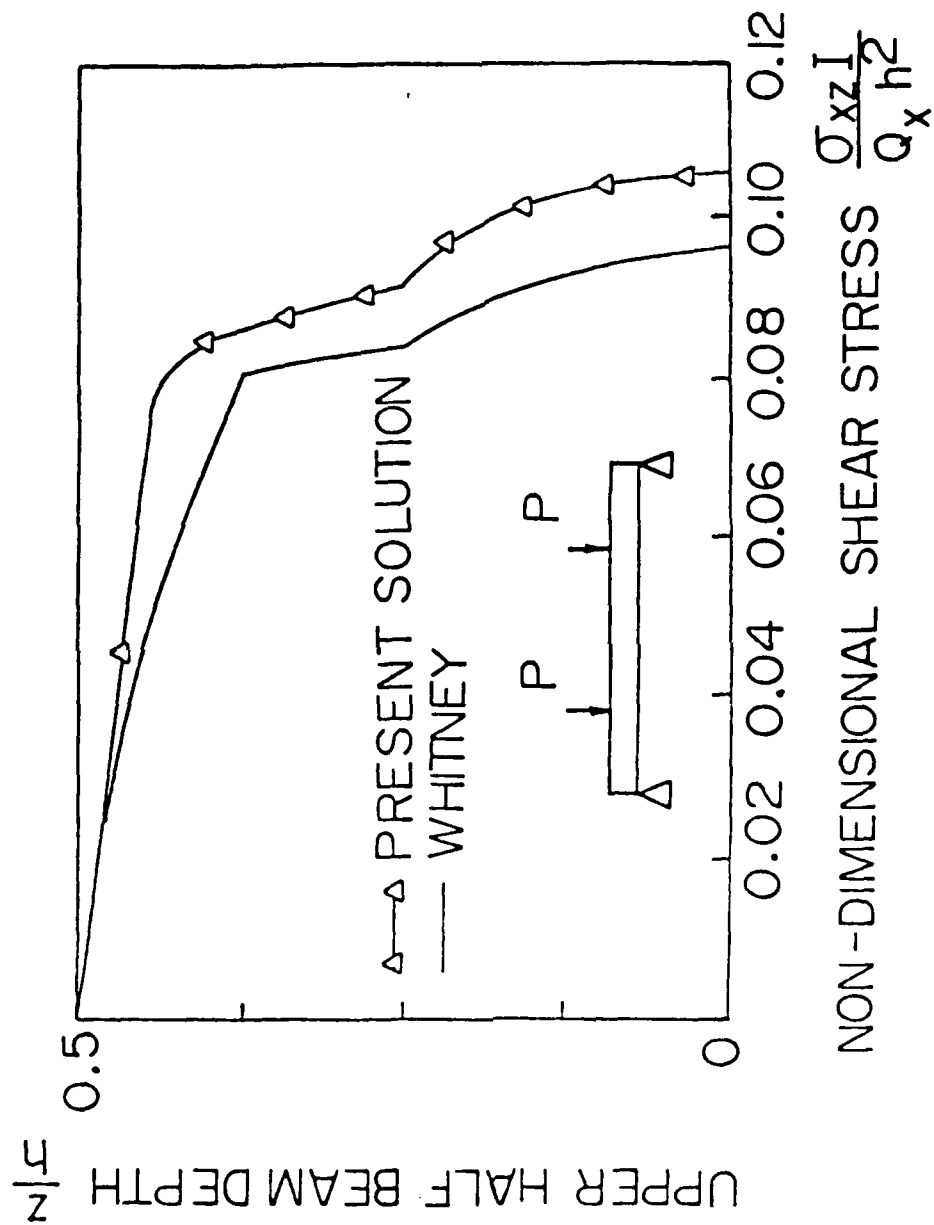


Figure 2.5 Distribution of transverse shear stress through the thickness of AS/4617 quasi-isotropic (0/90/±45)_s laminated beam under four point bending.

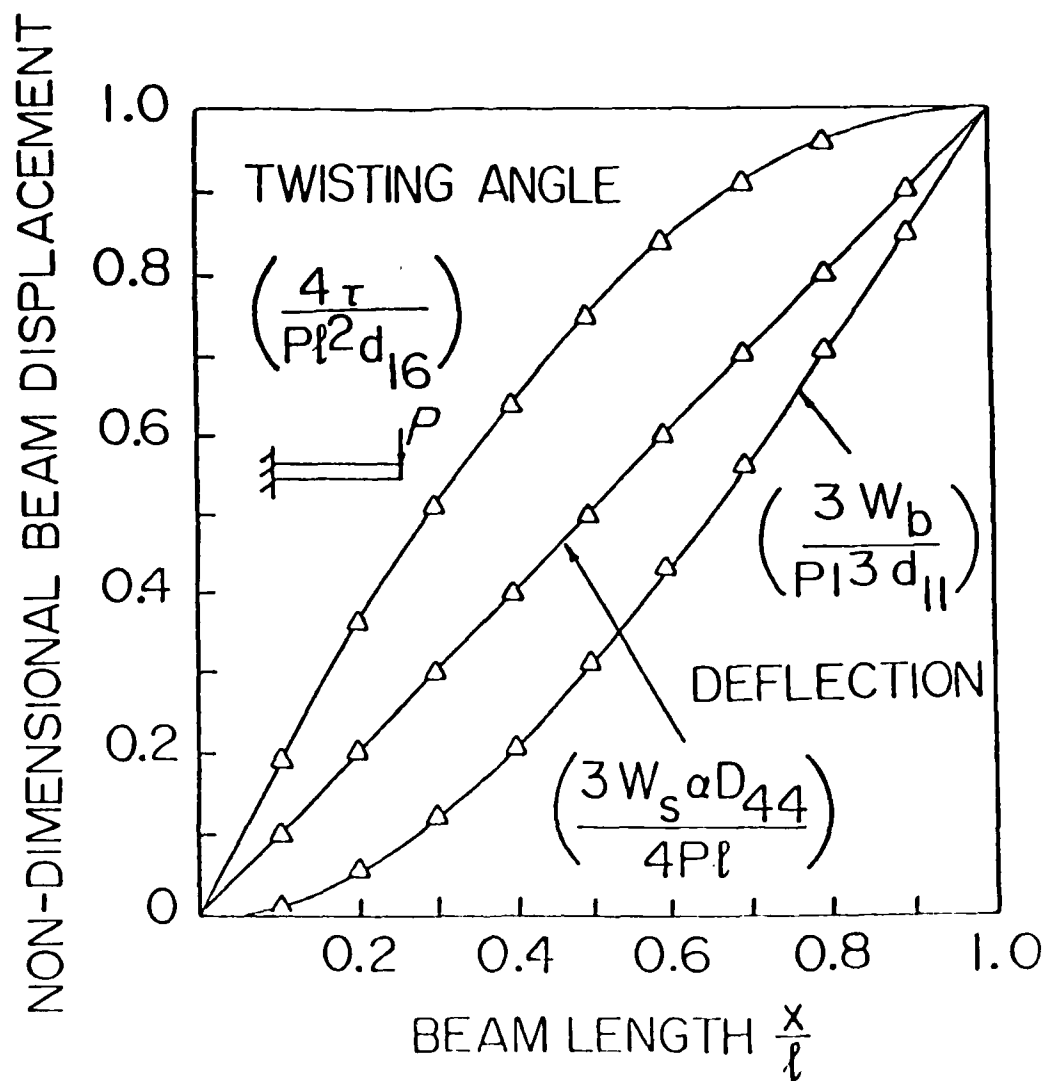


Figure 2.6 Distribution of deflections due to bending and shear deformation and twisting angle for a 16 ply $(45_4/-45_4)_s$ anisotropic laminated cantilever beam under end load P .

Table 2.3 Natural frequencies of an isotropic, homogeneous aluminum cantilever plate (wide beam) without shear deformation.

Mode	Natural Frequency (Hz)	
	Four Elements	Experimental Results [23]
First Bending	38.26	37.6
First Torsion	139.47	158.3
Second Bending	240.43	234.9
Second Torsion	418.42	518.8
Third Bending	673.21	658.1

3.2.2. Thin Anisotropic 6 Layer $[0_2/0]_s$ Laminated Cantilever Plates Without Shear Deformation

An anisotropic laminated plate with $[0_2/0]_s$ stacking sequences was analyzed. The results obtained using a variety of numbers of the present element are given in Table 2.4, for both consistent and lumped mass formulations. The experimental results used for comparison are given by Jensen [24]. The present solution is seen to converge at the four element level and to well-converge at the eight element level. The present solution is in good agreement with the experimental results for the bending frequencies. The percentage of discrepancies between the presently obtained torsional frequencies and the experimental values are within the approximate range of 10 - 20%. Again, this discrepancy may be due to the present modeling of the two-dimensional plate as a one-dimensional beam.

Table 2.4 Natural frequencies of a thin anisotropic 6 layer $[\theta_2/0]_s$ laminated cantilever plate (wide beam) without shear deformation.

Layup Sequence	Number of Elements												Exp. Results [24]
	Lumped Mass						Consistent Mass						
	1	2	4	8	10	16	1	2	4	8	10		
[0 ₂ /90] _s	7.7	9.9	10.7	10.9	11.0	10.9	11.1	11.0	11.0	11.0	11.0	11.2 B1	
	31.7	51.0 33.0	63.1 33.2	67.4 33.2	68.0 33.2	68.7 33.2	102.6 33.2	68.4 33.2	69.2 33.2	69.1 33.2	69.1 33.2	70.5 B2 42.4 T1	
[15 ₂ /0] _s	5.7	7.3	7.9	8.0	8.0	8.1	8.1	8.1	8.1	8.1	8.1	9.4 B1	
	41.0	36.1 44.9	49.2 40.2	51.6 40.8	51.9 40.9	52.3 41.0	84.7 43.3	52.8 41.1	52.6 41.0	52.5 41.1	52.5 41.0	66.2 B2 45.8 T1	
[30 ₂ /0] _s	3.9	5.0	5.5	5.6	5.6	5.6	5.7	5.6	5.6	5.6	5.6	6.6 B1	
	46.7	25.8 50.3	31.2 51.0	33.1 51.3	33.4 51.3	33.7 51.4	65.1 48.5	34.1 51.5	33.9 51.4	33.9 51.4	33.9 51.4	40.0 B2 59.1 T1	
[45 ₂ /0] _s	3.2	4.1	4.4	4.5	4.5	4.5	4.6	4.6	4.5	4.5	4.5	4.8 B1	
	42.1	21.0 44.5	25.7 44.9	27.4 44.9	27.7 44.9	28.0 45.0	49.6 43.0	28.4 45.0	28.2 45.0	28.1 45.0	28.1 45.0	29.8 B2 51.3 T1	
[60 ₂ /0] _s	2.8	3.9	4.0	4.0	4.0	4.0	4.1	4.1	4.1	4.1	4.1	4.3 B1	
	36.4	24.3 34.5	23.2 38.4	24.8 38.4	25.0 38.4	25.2 38.4	41.5 37.8	25.6 38.4	25.4 38.4	25.4 38.4	25.4 38.4	27.1 B2 47.7 T1	
[75 ₂ /0] _s	2.7	3.4	3.8	3.8	3.8	3.8	3.9	3.9	3.9	3.9	3.9	3.8 B1	
	32.9	17.6 33.1	22.0 34.5	23.6 34.5	23.8 34.5	24.0 34.5	34.4 38.4	24.4 34.5	24.2 34.5	24.2 34.5	24.2 34.5	25.1 B2 38.9 T1	
[90 ₂ /0] _s	2.6	3.9	3.7	3.8	3.8	3.8	3.8	3.8	3.8	3.8	3.8	3.7 B1	
	31.7	20.3 34.2	21.7 33.2	23.2 33.2	23.4 33.2	23.6 33.2	37.6 33.2	24.0 33.2	23.8 33.2	23.8 33.2	23.8 33.2	24.3 B2 38.2 T1	

B1, B2 and T1 mean respectively, first bending, second bending, and first torsional modes.

SECTION III

STATIC, DYNAMIC, AND BUCKLING FORMULATION OF A SYMMETRICALLY LAMINATED PLATE FINITE ELEMENT FOR A MICROCOMPUTER

1. Formulations

The beam conventions are given in Figure 1.1. The 8 d.o.f. beam finite element formulated previously is given in Figure 1.2.

The plate element described in Figure 1.3 was developed for the isotropic case by Cowper et al. The details of the derivations can be found in [15, 25] so they will not be presented here. Only those pertaining to the extension of the element to include the properties of composite materials are given here.

The strain energy expression can be written as follows for a symmetrically laminated composite plate

$$U_e = \frac{1}{2} \iint \begin{Bmatrix} W_{\xi\xi} \\ W_{\eta\eta} \\ 2W_{\xi\eta} \end{Bmatrix}^T \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} W_{\xi\xi} \\ W_{\eta\eta} \\ 2W_{\xi\eta} \end{Bmatrix} d\xi d\eta \quad (3.1)$$

where $W_{\xi\xi}$, $W_{\eta\eta}$, $2W_{\xi\eta}$ are the bending and twisting curvatures in local coordinates ξ and η of the elements, and D_{ij} are the flexural moduli. The integration is over the area of the triangular element.

2. Microcomputer Program

The formulations of the present 18 d.o.f. symmetrically laminated composite plate finite element has been coded into a microcomputer in FORTRAN language. The advantage of using FORTRAN language over BASIC are two-fold. The first is that the program can be developed more rapidly on a mainframe computer and then downloaded to the microcomputer. The second is that many algorithms already developed are in FORTRAN and these can be readily adapted and modified for use on the microcomputer. There now exist many commercial FORTRAN compilers for microcomputers.

Many algorithms already available have been adopted and modified for use on the microcomputer. Those algorithms that have been used in the plate program are presented here.

For static analysis, a banded [26] and a skyline [27] equation solver is used to minimize memory storage. The boundaries of the banded and skyline storage are illustrated in Figure 3.1. As an example, a comparison of various storage scheme for the stiffness matrix for a simply supported square isotropic plate under uniform load is given for 3 different meshes in Table 3.1. For eigenvalue analysis, a symmetric matrix eigenvalue solver is used to minimize memory storage. The Jacobi eigenvalue solver [28] has been modified to use a symmetric matrix storage scheme.

There are two methods of lumping masses. The first method, let it be called lumped mass method 1, can be found in Reference [29]. The total mass of the element is distributed in accordance with the distribution of the mass along the diagonal of the consistent mass matrix.

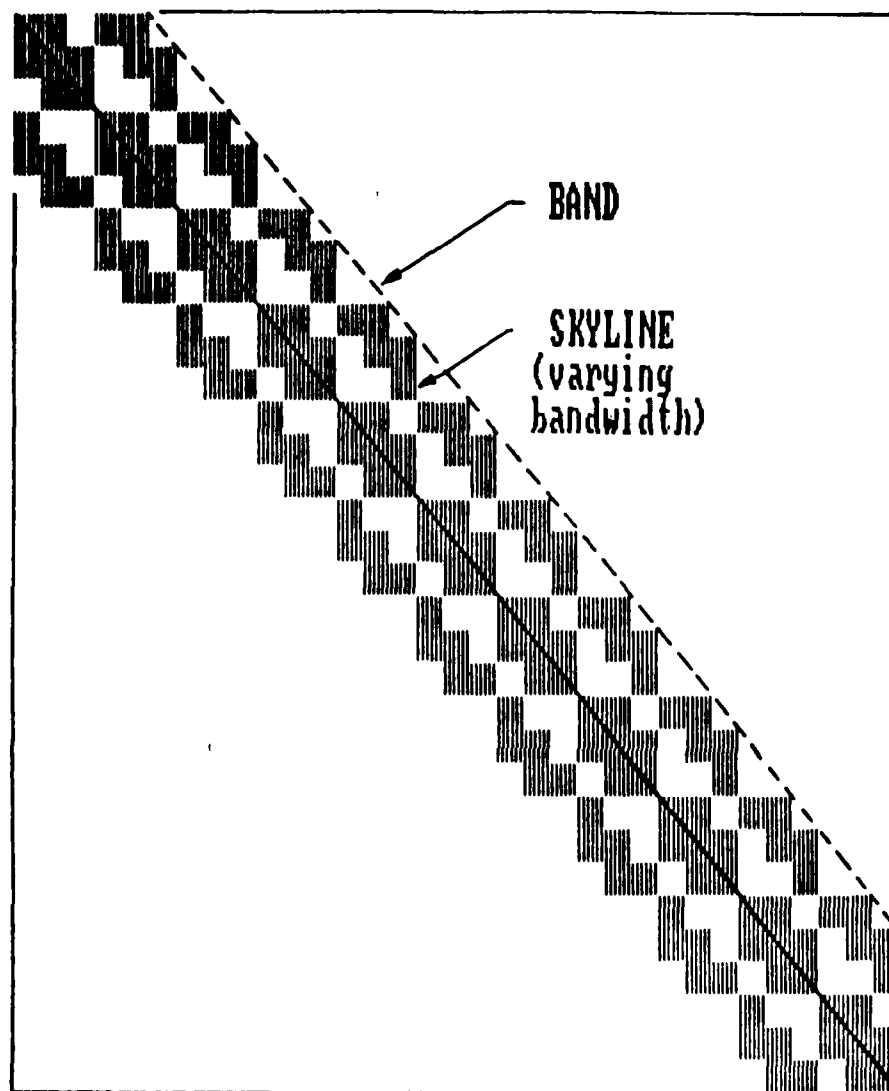


Figure 3.1 Various storage schemes.

Table 3.1 Various schemes for memory storage for [K]
for a simply supported square isotropic
plate under uniform load.

Mesh for a Quadrant	Degrees of Freedom	Storage Modes			
		Full Matrix	Symmetric	Band	Skyline
3x3	60	3600	1830	1395	1001
4x4	104	10816	5460	3569	2737
5x5	160	25600	12880	6664	5257

The second method is based on the most common method of lumping; let it be called lumped mass method 2. The total mass of the element is divided by 3 and the one-third masses becomes the terms m_{11} , m_{77} , and m_{13-13} , respectively. For the special case of lumped method two, a static condensation technique [30] eliminating the zero terms along the diagonal of the mass matrix is used to reduce the total number of the equations. The eigenvalue problem is given by

$$-\omega^2 \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} y_p \\ y_o \end{Bmatrix} + \begin{bmatrix} K_{pp} & K_{po} \\ K_{po}^T & K_{oo} \end{bmatrix} \begin{Bmatrix} y_p \\ y_o \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.2)$$

where $\{y_p\}$ denote the degrees of freedom associated with mass, $\{y_o\}$ denote the massless d.o.f.'s, and $[M]$ is the diagonal mass matrix with non-zero diagonal terms.

The massless d.o.f.'s $\{y_o\}$ can be expressed in terms of those associated with mass $\{y_p\}$ as follows.

$$[K_{po}]^T \{y_p\} = - [K_{oo}] \{y_o\} \quad (3.3)$$

Thus,

$$-\omega^2 [M] \{y_p\} + ([K_{pp}] - [K_{po}][K_{oo}]^{-1} [K_{po}]^T) \{y_p\} = 0 \quad (3.4)$$

or

$$-\omega^2[M]\{y_p\} + [K_{TT}]\{y_p\} = 0 \quad (3.5)$$

where $[K_{tt}]$ is a symmetrical reduced matrix. A technique making use of the diagonal matrix [31] which preserves the symmetry of the stiffness matrix and which allows use of a symmetric matrix eigenvalue solver is given below.

Given

$$[M] = \begin{bmatrix} a & & & & \\ & b & & & \\ & & c & & \\ & & & d & \\ & & & & e \\ & & & & & . \\ & & & & & & . \end{bmatrix} \quad (3.6)$$

and

$$[M]^{-1/2} = \begin{bmatrix} 1/\sqrt{a} & & & & \\ & 1/\sqrt{b} & & & \\ & & 1/\sqrt{c} & & \\ & & & 1/\sqrt{d} & \\ & & & & 1/\sqrt{e} \\ & & & & & . \\ & & & & & & . \end{bmatrix} \quad (3.7)$$

where $[M]$ is a diagonal matrix with nonzero diagonal terms, and $[M]^{-1/2}$ is a matrix with inverse square root diagonal terms,

then

$$[M]^{-1/2} [M] [M]^{-1/2} = [I] \quad (3.8)$$

and

$$-\lambda[I]\{y\} + [M]^{-1/2} [K_{TT}] [M]^{-1/2} \{y\} = 0 \quad (3.9)$$

where $[I]$ is the identity matrix. The stiffness matrix $[K_{TT}]$ premultiplied and postmultiplied by $[M]^{-1/2}$ results in a symmetric matrix.

For static analysis, the program computes the various displacements (transverse deflection, slopes, and bending and twisting curvatures) at the nodes. For eigenvalue analysis, the program computes the eigenvalues and the corresponding mode shapes. The program gives a three-dimensional view of the plate static deflection and plate vibration and buckling mode shapes.

3. EVALUATIVE ANALYSIS

The formulation, solution procedure, and program have been evaluated by performing the analysis of a series of example problems.

3.1. Static Analysis

Uniformly Loaded Cantilever Beam

In the first example, comparison is made for the centerline deflections and twist of two modelings for a uniformly loaded cantilever beam of aspect ratio of ten for various fiber angles. This comparison is made to further evaluate the previous beam element as to its accuracy and efficiency in various applications. Beam elements can be used to save computational time if its accuracy are close to those of the plate, but if deformation modes are required then plate elements have the advantage. The beam finite element has 4 d.o.f.'s at each of its two nodes: w_b and θ_b , the transverse deflection and its slope due to bending, and τ and τ' , the twisting angle and the rate

of twisting angle. As shown in Table 3.2, the plate model uses 40 elements and the beam model uses 5 elements. The dimensions of the beam are 10m in length by 1m in width by 0.0025m in thickness. The material properties of the ply are $E_x = 181 \text{ GP}_a$, $E_y = 10.3 \text{ GP}_a$, $\nu_{xy} = 0.28$, and $E_s = 7.17 \text{ GP}_a$. The centerline deflection at the tip and at 60% of the length from the fixed end are given in Figure 3.9. The results show a maximum discrepancy of 15% between the two modelings at a fiber angle of 30 degrees. For the centerline twist, the results are given in Table 3.3. These results show a maximum discrepancy of 15% at a fiber angle of 30 degrees. For the fiber angle of 30 degrees in this example, the coupling effect between bending and twisting is most pronounced as predicted by the shear coupling. It appears that the beam element does not predict the shear coupling well.

3.2. Vibration Analysis

3.2.1. Thin Anisotropic 6 Layer $[\theta_2/0]_s$ Cantilever Plate

The second example gives the natural frequencies of a thin anisotropic cantilever plate of aspect ratio of four. The dimensions and the material properties of the plate are given in Reference [32]. The results given in Table 3.4 using both the lumped mass and consistent mass formulation of the triangular plate elements are compared with those results given in Reference [32]. The first two columns are based on lumped mass method 1. The third and fourth columns are lumped mass method 2. The results show that the lumped mass method 1 does not seem to give improved results over those of lumped method 2. Furthermore, the three cases in increasing computational time and effort required are lumped mass method 2, lumped mass

Table 3.2 Comparison of centerline deflection of two modellings for a uniformly loaded cantilever beam.

Fiber Angle	Beam Deflection		Plate Deflection	
	Tip ($x=\ell$) (1)	$x=0.6\ell$ (2)	Tip ($x=\ell$) (3)	$x=0.6\ell$ (4)
0	5.45	2.57	5.299	2.518
30	34.24	16.17	29.108	13.351
45	58.89	27.81	52.788	24.550
60	79.37	37.48	74.578	35.113
90	95.69	45.19	93.160	44.262

Table 3.3 Comparison of centerline twist of two modelings for a uniformly loaded cantilever beam.

Fiber Angle	Beam Deflection		Plate Deflection	
	Tip ($x=l$) (1)	$x=0.6l$ (2) $x=0.6l/x=l$ (2)/(1)	Tip ($x=l$) (3)	$x=0.6l$ (4) $x=0.6l/x=l$ (4)/(3)
0	0.0	0.0	0.0	-----
30	-3.126	-2.885	-2.687	0.9240
45	-3.047	-2.813	-2.787	0.9317
60	-2.153	-1.987	-2.046	0.9347
90	0.0	0.0	0.0	-----

Table 3.4 Natural frequencies (Hz) for a thin anisotropic
6 layer $[\theta_2/0]_s$ cantilever plate.

Layup Sequence	Lumped Mass				Consistent Mass		Ref. [31]	
	8 el 50 dof	(1) 32 el 147 dof	8 el 8 dof	(2) 32 el 24 dof	50 dof	147 dof	Exp	365 dof
$[0_2/90]_s$	10.7	10.9	10.7	11.0	11.0	11.0	11.2	11.1
	21.0	31.0	22.9	32.2	39.5	39.5	42.4	39.5
	56.9	66.4	59.2	67.0	69.2	69.2	70.5	69.5
$[15_2/0]_s$	8.6	8.8	8.8	8.9	8.9	8.9	9.4	8.9
	24.7	35.2	26.7	36.7	42.9	42.8	45.8	42.9
	47.4	56.8	50.3	57.6	62.9	62.6	66.2	62.7
$[30_2/0]_s$	6.1	6.2	6.2	6.3	6.3	6.3	6.6	6.3
	27.2	34.5	28.9	35.6	37.3	37.2	40.0	37.3
	35.4	45.9	36.9	47.1	58.0	57.0	59.1	56.9
$[45_2/0]_s$	4.7	4.8	4.8	4.8	4.9	4.9	4.8	4.9
	25.3	28.8	27.4	29.4	30.1	30.0	29.8	30.1
	27.6	39.0	29.0	40.4	50.8	49.7	51.3	49.4
$[60_2/0]_s$	4.1	4.1	4.2	4.2	4.2	4.2	4.3	4.2
	22.7	25.1	23.5	25.5	26.1	26.0	27.1	26.1
	22.9	32.8	24.9	24.1	42.8	41.9	47.7	41.7
$[75_2/0]_s$	3.8	3.8	3.8	3.9	3.9	3.9	3.8	3.9
	19.8	23.5	20.9	23.7	24.3	24.3	25.1	24.3
	21.5	28.9	22.9	30.1	37.3	36.8	38.9	36.7
$[90_2/0]_s$	3.7	3.8	3.7	3.8	3.8	3.8	3.7	3.8
	18.7	23.0	19.9	23.2	23.8	23.8	24.3	23.9
	21.1	27.6	22.3	28.7	35.2	35.1	38.2	35.1

method 1, and consistent mass method. The lumped mass method 2 uses fewer degrees of freedom. Thus one can take advantage of the condensation of the mass matrices as described in Equations 3.6-3.9. The results using consistent mass formulation are given in the third and fourth columns. All the results compare very well with the experimental and finite element (8 node, 40 d.o.f., quadrilateral, mixed, hybrid element) results given in Reference [32].

3.2.2. Anisotropic 6 Layer $[\theta_2/0]_s$ Cantilever Beam

The third example given in Table 3.5 is a comparison of two modelings of an anisotropic cantilever beam with an aspect ratio of ten. The material properties are given in Reference [32]. The dimensions are 10m in length, 1m in width, and 0.804mm in thickness. The plate modeling uses both the 20 (1x10 mesh) and 40 (2x10 mesh) elements, respectively, while the beam modeling uses 4 and 10 elements, respectively.

It is seen that the three sets of results obtained by using the three formulations (1. Beam element with consistent mass, 2. plate element with lumped mass, 3. plate element with consistent mass), each of which using two selected meshes, are in good agreement. The maximum discrepancies occur in the case with fiber angle of 30 degrees where the bending and twisting moments are more pronouncedly coupled. In that case, the discrepancy in natural frequency of the third mode between the 10 elements and 40 plate elements (both with consistent mass formulations) is only 5%.

Table 3.5 Comparison of natural frequencies (Hz) of two modelings for a cantilever beam.

Layup Sequence	Beam Element		Plate Element					
	Consistent Mass 4 el 17 dof	Consistent Mass 10 el 52 dof	Lumped Mass		Consistent Mass			
			20 el 20 dof	40 el 30 dof	20 el 122 dof	40 el 183 dof		
[0 ₂ /90]s	0.01026	0.01026	0.01022	0.01022	0.01026	0.01026		
	0.06436	0.06429	0.04751	0.06301	0.06432	0.06432		
	0.07716	0.07716	0.06322	0.06755	0.08235	0.08234		
[30 ₂ /0]s	0.00525	0.00525	0.00549	0.00549	0.00551	0.00551		
	0.03278	0.03274	0.03374	0.03390	0.03440	0.03439		
	0.09172	0.09092	0.06970	0.09393	0.09651	0.09646		
[45 ₂ /0]s	0.00424	0.00424	0.00435	0.00435	0.00436	0.00436		
	0.02655	0.02651	0.02688	0.02690	0.02730	0.02730		
	0.07456	0.07404	0.06260	0.07455	0.07645	0.07645		
[60 ₂ /0]s	0.00378	0.00378	0.00382	0.00382	0.00383	0.00383		
	0.02374	0.02371	0.02363	0.02364	0.02399	0.02400		
	0.06685	0.06634	0.05351	0.06548	0.06715	0.06717		
[90 ₂ /0]s	0.00353	0.00353	0.00352	0.00352	0.00354	0.00354		
	0.02218	0.02215	0.02180	0.02181	0.02216	0.02216		
	0.06250	0.06204	0.04560	0.06023	0.06204	0.06204		

3.3. Buckling Analysis

Simply Supported Anisotropic Square Plate Under Compressive Axial Load.

The dimensions and stiffness properties of the present plate example are given in Reference [33]. The critical buckling loads obtained for various fiber orientations using 32 triangular elements (4x4 rectangular mesh) are shown in Table 3.6. The results obtained by using the Galerkin method [33], and the experimental results [34], and the exact solutions for the orthotropic case only [35] are all shown for comparison. It is seen that for the cases with fiber angles of 0 and 90 degrees, the present results agree well with the exact solution [35]. For the case with fiber angles of 30, 45, and 60 degrees, the present results are in fair agreement with those given by [34, 35].

Table 3.6 Critical buckling loads for a simply supported anisotropic square plate under uniform compressive axial load.

Fiber Angle	Present 32 el 94 dof	Ref. [33]	Exp. [34]	Exact [35]
0	318.86	285	271	318.91
30	387.90	425	399	--
45	354.67	406	364	--
60	355.40	381	433	--
90	203.88	210	251	203.75

SECTION IV
A 36 DOF SYMMETRICALLY LAMINATED TRIANGULAR ELEMENT
WITH SHEAR DEFORMATION AND ROTATORY INERTIA

1. Formulation

The effect of transverse shear deformation is incorporated into the formulations by expressing the strain energy of deformation as the sum of the thin plate flexural energy and the energy of transverse shear deformation.

$$U = U_b + U_s \quad . \quad (4.1)$$

The effects of rotatory inertia is included in the formulation of the kinetic energy expression.

For the present plate finite element, two displacement functions are used which include the displacements due to bending W_b and that due to transverse shear deformation, W_s . The total displacement is

$$W_t = W_b + W_s \quad (4.2)$$

Strain energy can be expressed as follows

$$\begin{aligned}
U = & \frac{1}{2} \iint \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix}^T [D] \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} dx dy \\
& + \frac{1}{2} \iint \begin{Bmatrix} \mu_x \\ \mu_y \end{Bmatrix}^T [A] \begin{Bmatrix} \mu_x \\ \mu_y \end{Bmatrix} dx dy
\end{aligned} \tag{4.3}$$

where the curvatures are given as

$$k_x = \partial^2 W_b / \partial x^2, \quad k_y = \partial^2 W_b / \partial y^2, \quad \text{and} \quad k_{xy} = 2\partial^2 W_b / \partial x \partial y \tag{4.4}$$

and the shearing angles are given as

$$\mu_x = \partial W_s / \partial x \quad \text{and} \quad \mu_y = \partial W_s / \partial y \tag{4.5}$$

The matrices [D] and [A] contains the flexural and shearing moduli, respectively.

It is seen that the flexural energy is only in terms of the displacement due to bending W_b and that the shear strain energy is only in terms of the displacement due to shear deformation W_s .

The kinetic energy including the effect of rotatory inertia is given by

$$\begin{aligned}
T = & \rho \left[\frac{h}{2} \iint \dot{W}_t^2(x,y) dx dy \right. \\
& \left. + \frac{1}{2} \iint [\dot{W}_{b,x}^2 + \dot{W}_{b,y}^2] dx dy \right]
\end{aligned} \tag{4.6}$$

where ρ is the density per unit volume, W_t is the total deflection, $W_{b,x}$ and $W_{b,y}$ are the first derivative of the deflection due to bending alone, and h , and I are thickness, and moment of inertia, respectively.

The displacement functions for the present 36 d.o.f. triangular element (see Figure 4.1) as described in local coordinates are

$$W_b = \sum_{i=1}^{20} a_i \xi^{m_i} \eta^{n_i} m_i, \quad n_i = 0 \text{ for } i > 20 \quad (4.7)$$

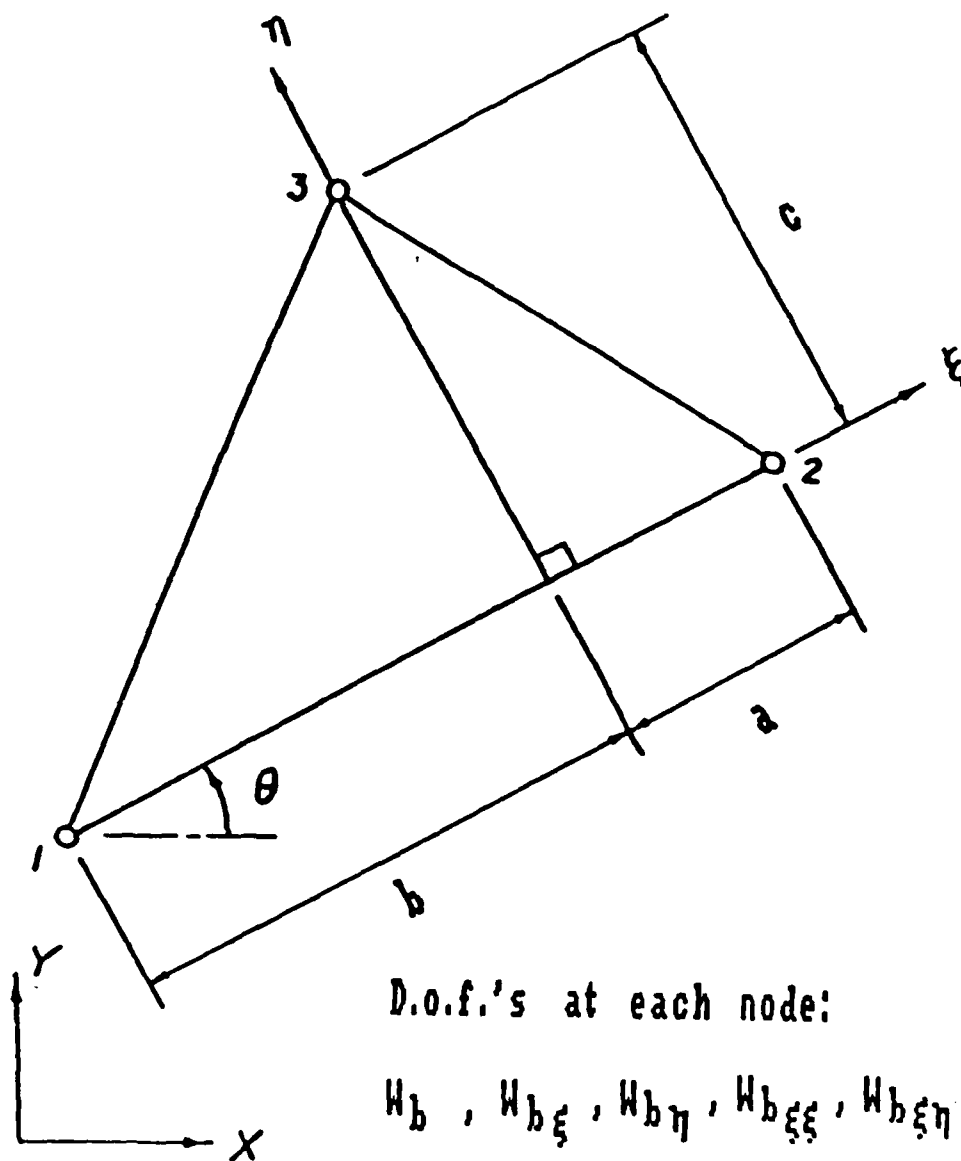
$$W_s = \sum_{i=21}^{40} a_i \xi^{r_i} \eta^{s_i} r_i, \quad s_i = 0 \text{ for } i < 21$$

where the displacement due to bending W_b is described in Reference [25]. The displacement due to shear deformation W_s is assumed to be of the same form as that for W_b . Thus these displacement functions are independent of each other.

The derivation of the stiffness matrix due to bending alone has been derived previously for the isotropic case by Cowper et al. [25] and extended to the anisotropic case by the present authors [36] for the case of thin plate. All the explanations and derivations given in [25] and [26] will not be repeated here. Only the derivation of the stiffness matrix due to shear deformation alone is given here.

The strain energy expression in local coordinates due to shear deformation can be expressed as

$$U_s = \frac{1}{2} \iint \begin{Bmatrix} W_{s\xi} \\ W_{s\eta} \end{Bmatrix}^T \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} W_{s\xi} \\ W_{s\eta} \end{Bmatrix} d\xi d\eta \quad (4.8)$$



D.o.f.'s at each node:

w_b , $w_{b\xi}$, $w_{b\eta}$, $w_{b\xi\xi}$, $w_{b\xi\eta}$, $w_{b\eta\eta}$

w_s , $w_{s\xi}$, $w_{s\eta}$, $w_{s\xi\xi}$, $w_{s\xi\eta}$, $w_{s\eta\eta}$

Figure 4.1 36 d.o.f. triangular plate element.

where W_{s_ξ} and W_{s_η} are the shear angles in local coordinates ξ and η of the elements, and A_i are the shear moduli. The integration is over the area of the triangular element. The expression in terms of global coordinates follows the same procedure for transformation and rotation of the local coordinates as described in [2].

Shear correction factor α_1 and α_2 are incorporated into the shear moduli as follows $A_{44} = \alpha_1 G_{yz} h$, and $A_{55} = \alpha_2 G_{yz} h$, where $G_{yz} = G_{xz}$ is assumed for the shear stiffnesses. They are assumed to be 5/6 unless otherwise specified. This value was used by Reissner [16], and Whitney [37].

The equations of motion for the case of free vibration for the element can be expressed in matrix form as

$$-\omega^2 \begin{bmatrix} M_s & M \\ M & M \end{bmatrix} \begin{Bmatrix} W_b \\ W_s \end{Bmatrix} + \begin{bmatrix} K_{bb} & 0 \\ 0 & K_{ss} \end{bmatrix} \begin{Bmatrix} W_b \\ W_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4.9)$$

where

$[M_s]$ = consistent mass and matrix including the effect of rotatory inertia matrix,

$[M]$ = consistent mass matrix,

$[K_{bb}]$ = bending stiffness matrices,

$[K_{ss}]$ = shear stiffness matrices,

$[W_b]$ = degrees of freedom corresponding to the displacement due to bending, and

$[W_s]$ = degrees of freedom corresponding to the displacement due to shear deformation, and

ω = natural frequency.

2. Evaluative Analysis

In order to evaluate the performance of the present finite element formulation, and to find an indication of the range of applicability, a series of symmetrically chosen examples was analyzed and compared with various alternative solutions.

2.1 Isotropic Simply-Supported Square Thick Plates

The first example chosen was to find several lower mode natural frequencies for an isotropic simply supported square plate. For this example, an exact 3D elasticity solution [38] as well as solution by Mindlin theory [39] are available for comparison. The results for the nondimensional lower mode natural frequencies for various thickness-to-length ratio (h/a) obtained using a 5x5 mesh of the present 36 d.o.f. element are given in Table 4.1. The natural frequencies were nondimensionalized as

$$\Omega = \rho \omega^2 a^4 / \pi^4 D \quad (4.10)$$

where ρ = density (mass per unit volume), ω = frequency (radians/sec), a = side length, and D = the bending rigidity of the plate.

An important parameter in the thick plate analysis is the shear coefficient α . In [40], Mindlin suggested a value of α to be $\pi^2/12$. In [16] and [37], a value of 5/6 was used. When obtaining the results of Table 1, a value of 5/6 was thus used for α .

Table 4.1 Nondimensional frequency Ω for various thicknesses and modes for a simply supported isotropic square thick plate.

Thickness ----- Length	Mode	Present 32 Elements	Mindlin [39]	Thin Plate Theory	3D Elasticity [39]
		Ω	$\% \text{diff-}$ erence^*	Ω	$\% \text{diff-}$ erence^*
0.05	(2,1)	23.924	0.14	23.909	0.20
0.05	(2,2)	59.736	0.21	59.736	0.21
0.10	(1,1)	3.731	0.26	3.729	0.32
0.10	(2,1)	21.243	0.57	21.204	0.75
0.10	(2,2)	50.039	0.78	49.896	1.06
0.10	(3,1)	74.428	0.80	74.044	1.31
0.10	(3,2)	117.253	0.93	116.472	1.59
0.10	(3,3)	202.548	1.02	200.595	1.98
0.10	(4,2)	243.704	-0.19	238.105	2.12
0.20	(1,1)	3.126	0.88	3.119	1.10
0.20	(2,1)	14.946	1.69	14.882	2.12
0.20	(2,2)	31.238	2.24	31.055	2.81
				25.001	-4.36
				64.003	-6.92
				4.000	-6.92
				24.999	-17.01
				64.000	-26.90
				100.001	-33.29
				168.998	-42.80
				324.002	-58.32
				399.996	-64.44
				4.000	-26.85
				25.000	-64.44
				64.002	-100.30
					23.958
					59.860
					3.741
					21.365
					50.432
					75.026
					118.349
					204.644
					243.252
					3.153
					15.203
					31.953

★ percentage difference as compared to 3-D elasticity solution [39].

It is seen in Table 4.1 that the present results agree with the 3D elasticity solution quite well for practically all the modes shown with thickness-to-length ratio (h/a) up to 0.2. The maximum discrepancy is 2.24%. The classical thin plate solution is also shown in order to indicate the effect of shear deformation and rotatory inertia at the various h/a level.

Table 4.2 shows the first mode nondimensional frequencies for the same isotropic simply supported square thick plate but with more values of h/a and the results are compared with those given by Rao, Venkataramana, and Raju [41].

2.2. Isotropic Clamped Square Thick Plates

The boundary conditions were then changed from clamped to simply-supported and the results are given in Table 4.3.

The results by Rao et al. [41] was based on a 36 d.o.f. isotropic triangular plate element originated by Cowper, Lindberg, and Olson [25] and Bell [42]. That element was formulated on the basis of Reissner's thick plate theory. The 36 d.o.f.'s include the original 18 for bending (see Figure 4.1) and the additional six at each of the three corner nodes:

$$\psi_{\xi}, \partial\psi_{\xi}, \partial\psi_{\xi}/\partial\eta$$

$$\partial\psi_{\eta}, \partial\psi_{\eta}/\partial\xi, \partial\psi_{\eta}/\partial\eta$$

where ψ_{ξ} and ψ_{η} are defined as the total slope due to bending and shear deformation in the ξz and ηz planes, respectively. It is noted that

Table 4.2 First mode nondimensional frequencies of an isotropic simply supported square thick plate.

Thickness ----- Length	8 element	16 element	32 element	%diff- erence*	Rao et al. [41]
0.001	4.002	4.000	4.000	0.00	4.000
0.050	3.930	3.928	3.928	2.23	3.840
0.100	3.733	3.731	3.731	5.53	3.525
0.150	3.410	3.408	3.407	6.57	3.184
0.200	3.127	3.125	3.125	8.77	2.851
0.250	2.798	2.797	2.797	9.14	2.541

* percentage difference as compared to the results by Rao et al.[41].

the present 36 d.o.f. elements include the 18 for bending and the additional six at each of the three corners: (see Figure 4.1)

$$W_s, W_{s_\xi}, W_{s_{\xi\xi}}, W_{s_{\xi\eta}}, W_{s_{\eta\eta}}$$

It is noted that for the results given in Table 4.2 and 4.3, the shear coefficient α was assumed as 1.0 in [41] but it was still assumed as 5/6 in this study in order to be consistent with that used in [16], [40] and Table 1.

The solution of [41] was obtained based on a 4x4 mesh with 32 elements. It is seen in both tables that the present solutions has converged at the 16 element level. The percentage differences between the present results and those of [41] increase as the thickness-to-length ratio increases and it reaches 9.14% at $h/a = 0.25$ for the simply supported plate and -8.32% at $h/a = 0.25$ for the clamped plate.

2.3. Single Layer Orthotropic Simply Supported Plates of Various Different Materials

Four different kinds of orthotropic materials were used in the analysis. The values for the various moduli of elasticity and shear moduli are all given in Table 4.4. The lowest natural frequencies for simply supported orthotropic homogeneous square plates using the different four materials and with various thickness-to-length ratios (h/a) using 50 elements are given in Table 4.5. The present results for various thickness-to-length ratios (h/a) using 50 elements are compared with those by Bhimarraddi and Stevens [43] using a higher

Table 4.3 First mode nondimensional frequencies of an isotropic clamped square thick plate.

Thickness ----- Length	8 element	16 element	32 element	%diff- erence	Rao et al. [41]
0.001	14.312	13.380	13.308	0.02	13.305
0.050	13.589	12.740	12.673	-0.31	12.712
0.100	11.817	11.156	11.104	-1.39	11.258
0.150	9.589	9.125	9.088	-4.83	9.526
0.200	7.814	7.502	7.476	-5.68	7.901
0.250	6.246	6.039	6.022	-8.32	6.523

* percentage difference as compared to the results by Rao et al.[41].

Table 4.4 Orthotropic material properties for example plates.

Material	I	II	III	IV
Ex (GPa)	55.8979	36.0885	35.2059	17.5539
Ey (GPa)	13.7293	26.2818	28.7335	12.8467
ν_{xy}	0.2770	0.1050	0.1770	0.1500
Gxy (GPa)	5.5898	4.9033	7.4531	2.7459
Gyz (GPa)	4.9033	4.0208	6.1782	2.3536
Ex/Ey	4.07	1.37	1.23	1.37
Gxy/Ey	0.41	0.19	0.26	0.21
Gyz/Ey	0.36	0.15	0.22	0.18

Table 4.5 Comparison of lowest natural frequency values $\Omega(\omega/\sqrt{E_x/\rho h^2})$ for simply supported orthotropic homogeneous square plates.

Thickness ----- Length	Solution Method*	Material I Ω Xdiff- erence*	Material II Ω Xdiff- erence	Material III Ω Xdiff- erence	Material IV Ω Xdiff- erence
0.1	1	0.03626	0.04200	0.04653	0.04367
	2	0.03617	0.04135	0.04707	0.04362
	3	0.03800	0.04330	0.04891	0.04562
	4	0.03615	0.04133	0.04705	0.04359
0.2	1	0.12708	0.14638	0.16568	0.15383
	2	0.12628	0.14531	0.16800	0.15362
	3	0.14844	0.16914	0.19107	0.17818
	4	0.12596	0.14496	0.16767	0.15328
0.3	1	0.24408	0.27971	0.32306	0.29689
	2	0.24226	0.28005	0.32861	0.29678
	3	0.32181	0.36668	0.41421	0.38628
	4	0.24098	0.27866	0.32727	0.29536
0.4	1	0.37072	0.42316	0.49658	0.45262
	2	0.36885	0.42714	0.50697	0.45354
	3	0.54541	0.62145	0.70200	0.65468
	4	0.36561	0.42354	0.50347	0.44984
0.5	1	0.50001	0.56908	0.67583	0.61217
	2	0.50021	0.57898	0.69285	0.61566
	3	0.80626	0.91867	1.03975	0.96780
	4	0.49366	0.57160	0.68567	0.60807

- * 1 - Present method with 50 elements,
 2 - Higher order theory [43],
 3 - Classical Thin Plate Theory, and
 4 - Generalized Reissner-Mindlin shear deformation theory [44].

* Percentage difference as compared to the results by the generalized Reissner-Mindlin shear deformation theory [44].

order plate theory, Yang et al. [44] using a generalized Reissner-Mindlin plate theory, and the classical thin plate theory. The maximum discrepancy between the present results and those using Reissner-Mindlin Theory is 1.6%.

The percentage discrepancy for those obtained using the thin plate theory are given to indicate the extent of the effect of shear deformation and rotatory inertia at the various h/a ratios.

2.4. Three Ply (0/90/0) Orthotropic Simply-Supported Square Plates

In Table 4.6, a comparison of the lowest natural frequency values ($\Omega/\sqrt{E_x/\rho h^2}$ for a three-ply (0/90/0) simply supported laminated square plate using material I in Table 4.4 is given. The results for various thickness-to-length (h/a) and beta values (β = ratio of middle-to-outer ply thickness = $h_2 / h_1 = h_2 / h_3$; h_1 , h_2 , and h_3 are respectively the thicknesses of the three plies) are compared with those obtained by Bhimaraddi and Stevens [43] using a higher order theory, by Yang, Norris, and Stavsky [44], and classical thin plate theory. It is seen that the present results are in consistent agreement with those by [44] based on a generalized Reissner-Mindlin shear deformation theory with all the h/a and β values considered. However, it is seen that the percentage difference between the present results (or the solution of [44]) and those from higher order theory [43] increases as h/a and β increase. This discrepancy reaches an extreme at $h/a = 0.5$ and $\beta = 20$.

2.5. Five Ply (0/90/0/90/0) Orthotropic Clamped Square Plates

The natural frequencies for the first five modes for a five ply (0/90/0/90/0) orthotropic clamped square plate were obtained using

Table 4.6 Comparison of lowest natural frequency values $\Omega(\omega/\sqrt{E_x/\rho h^2})$ for three-ply (0/90/0) laminated square simply supported plates.

Thickness ----- Length	Solution Method*	1		5		10		15		20	
		Ω	Δ diff- erence*	Ω	Δ diff- erence*	Ω	Δ diff- erence*	Ω	Δ diff- erence*	Ω	Δ diff- erence*
0.1	1	0.072511	-0.5	0.072511	-0.9	0.072511	-1.0	0.072511	-0.9	0.072511	-0.8
	2	0.073040	0.2	0.074240	1.4	0.074580	1.9	0.074670	2.1	0.074710	2.2
	3	0.076670	5.2	0.076590	4.6	0.076540	4.5	0.076520	4.6	0.076500	4.4
	4	0.072900	--	0.073190	--	0.073210	--	0.073150	--	0.073100	--
0.2	1	0.254156	0.1	0.254156	-1.3	0.254156	-1.5	0.254156	-1.3	0.254156	-1.1
	2	0.255520	0.6	0.269070	4.3	0.273480	5.7	0.274760	6.3	0.275290	6.6
	3	0.299490	15.2	0.299190	14.0	0.298990	13.7	0.298890	13.8	0.298830	14.0
	4	0.253960	--	0.257440	--	0.257920	--	0.257510	--	0.257050	--
0.3	1	0.488163	0.5	0.488163	-1.6	0.488163	-1.9	0.488163	-1.7	0.488163	-1.5
	2	0.491190	1.1	0.534380	7.7	0.550000	10.5	0.554700	11.7	0.556650	12.4
	3	0.649270	33.6	0.648610	30.7	0.648190	30.2	0.647970	30.5	0.647840	30.8
	4	0.485880	--	0.496310	--	0.497780	--	0.496690	--	0.495410	--
0.4	1	0.741440	0.5	0.741440	-2.0	0.741440	-2.3	0.741440	-2.1	0.741440	-1.8
	2	0.749060	1.6	0.834480	10.3	0.868500	14.4	0.879320	16.1	0.883400	17.0
	3	1.100190	49.2	1.098280	45.3	1.098560	44.8	1.098620	45.1	1.097970	45.5
	4	0.737400	--	0.756320	--	0.758910	--	0.757290	--	0.754840	--
0.5	1	1.000023	0.4	1.000023	-2.2	1.000024	-2.6	1.000024	-2.3	1.000023	-2.0
	2	1.016920	2.1	1.150560	12.5	1.208750	17.8	1.227380	19.9	1.235010	21.0
	3	1.626670	63.3	1.625040	58.9	1.623970	58.2	1.623420	58.5	1.623100	59.0
	4	0.996070	--	1.022940	--	1.026470	--	1.023950	--	1.020920	--

- * 1 - Present method with 50 elements,
 2 - Higher order theory [43],
 3 - Classical Thin Plate Theory, and
 4 - Generalized Reissner-Mindlin shear deformation theory [44]

* percentage difference as compared to the results by the generalized Reissner-Mindlin shear deformation theory [44].

$\rho = h_2 / h_1 = h_2 / h_3$ = ratio of the 90 degree ply to 0 degree ply where h_1, h_2, h_3 are the respective ply thickness of the 3 layer laminate.

three different meshes of the present 36 d.o.f. plate elements for two different thickness/length ratio ($h/a = 0.01$ and 0.1 , respectively). The results are given in Table 4.7 along with those by Craig and Dawe [45] for comparison. The nondimensional frequency Ω is defined as

$$\Omega = \frac{\omega a^2}{h} \left[\frac{\rho}{Q_{11}} \right]^{1/2} \quad (4.11)$$

where Q_{11} is the inplane stiffness for the laminate. The present analysis was based on the same values of the shear coefficients ($\alpha_1 = 0.87323$ and $\alpha_2 = 0.59139$) as used in [45]. The same material properties were also used: $E_x/E_y = 30$, $G_{xy}/E_y = 0.6$, $G_{xz}/E_y = 0.5$, and Poisson's ratio $\nu_{xy} = 0.25$. Each 0 degree ply is two-thirds the thickness of the 90 degree ply. In [45], a finite strip formulation was developed and a Raleigh Ritz method was used to validate the finite strip formulation. The results given in [45] was based on the Raleigh-Ritz method using 5 term series and three finite strips using 2 term series. The results using classical plate theory were also given in Table 4.7 to validate the accuracy of the present solution in the special case of thin plates.

The percentage differences between the present solution and those by Raleigh Ritz method and finite strip formulation are with 2.14% in all cases.

2.6. Five Ply (0/90/0/90/0) Orthotropic Simply Supported Square Plates

The boundary conditions were then changed from clamped to simply-supported and the results are given in Table 4.8 along with Craig and

Table 4.7 Nondimensional frequency Ω for an orthotropic
(0/90/0/90/0) clamped plate.

h/a	Mode Number	Classical Plate Theory	Raleigh Ritz [45]	Finite Strip [45]	8 element element	16 element element	32 element element	%diff- erence*
0.01	(1,1)	9.26			9.453	9.197	9.182	0.84
	(1,2)	16.13			17.135	16.096	15.960	1.05
	(2,1)	21.79			22.687	21.564	21.436	1.62
	(2,2)	25.83			31.392	26.229	25.522	1.19
	(1,3)	28.64			32.511	28.701	28.186	1.59
0.1	(1,1)		5.637	5.637	5.577	5.520	5.516	2.14
	(1,2)		8.582	8.582	8.823	8.652	8.627	-0.54
	(2,1)		9.971	9.971	10.281	10.153	10.136	-1.65
	(2,2)		11.931	11.931	12.804	12.217	12.134	-1.70
	(1,3)		12.627	12.627	12.873	12.530	12.481	1.15

* percentage difference between present 32 elements solution and those
by classical thin plate theory for length-to-length ratio (h/a) = 0.01,
and those by finite strip method [45] for h/a = 0.1.

Dawe [14] for comparison. The results given in [45] were based on the Raleigh-Ritz method using 5 term series and three finite strips using 2 term series. The results using classical plate theory were also given in Table 4.8 to validate the accuracy of the present solution in the special case of thin plates.

The percentage differences between the present solution and those by Raleigh Ritz method and finite strip formulation are with 2.30% in all cases.

2.7. Anisotropic ($\theta = 30$ degrees) Clamped Square Plate

Finally, an example of anisotropic (with fiber angle $\theta = 30$ degrees) clamped square plate was analyzed. The present analysis was based on the same values of the shear coefficients ($\alpha_1 = \alpha_2 = 5/6$) as used in [45]. The same material properties were also used: $E_x/E_y = 10$, $G_{xy}/E_y = 0.25$, $G_{xz}/E_y = 0.25$, and Poisson's ratio $\nu_{xy} = 0.3$. The nondimensional frequency α is defined as

$$\alpha = \frac{a^2}{h} \left[\frac{12(1-\nu_{xy}\nu_{yx})}{E_x} \right]^{1/2} \quad (4.12)$$

The results for the first four mode nondimensional natural frequencies α for this plate were obtained using various meshes of the present elements for two different thickness-to-length ratios ($h/a = 0.01$ and 0.1). The present results and those obtained in [45] (using 3 finite strips and a 6-term series) are both shown in Table 4.9 for comparison. The results show a discrepancy of about 8.6% for the thickness-to-length ratio (h/a) of 0.1 between the present and the finite strip

Table 4.8 Nondimensional frequency Ω for an orthotropic (0/90/0/90/0) simply-supported plate.

h/a	Mode Number	Classical Plate Theory	Raleigh Ritz [45]	Finite Strip [45]	8 element	16 element	32 element	% difference*
0.01	(1,1)	4.214			4.207	4.205	4.205	0.21
	(1,2)	9.829			9.975	9.796	9.784	0.46
	(2,1)	13.840			13.926	13.751	13.738	0.74
	(2,2)	16.854			17.462	16.767	16.729	0.74
	(1,3)	20.646			21.674	20.632	20.455	0.93
0.1	(1,1)		3.604	3.604	3.570	3.569	3.569	0.97
	(1,2)		7.093	7.093	7.098	7.026	7.021	1.01
	(2,1)		8.838	8.838	8.822	8.770	8.766	0.81
	(2,2)		10.799	10.799	10.763	10.562	10.551	2.30
	(1,3)		11.714	11.715	11.839	11.602	11.564	1.29

* percentage difference between present 32 elements solution and those by classical thin plate theory for length-to-length ratio (h/a) = 0.01, and those by finite strip method [45] for h/a = 0.1.

Table 4.9 Nondimensional frequency Ω for an anisotropic
(30 degrees) clamped plate.

h/a	Mode Number	8 element	16 element	32 element	50 element	Zdiff- erence*	Finite Strip [45]
0.01	1	21.488	21.306	21.259	21.247	0.13	21.22
	2	34.984	33.185	33.066	33.019	-0.49	33.18
	3	53.217	51.282	50.523	50.389	-0.08	50.43
	4	55.222	51.556	51.362	51.297	0.17	51.21
0.1	1	15.531	15.463	15.447	15.443	7.53	14.28
	2	24.953	24.250	24.199	24.182	8.03	22.24
	3	29.520	29.199	29.167	29.157	8.63	26.64
	4	35.293	33.813	33.535	33.492	7.41	31.01

* percentage difference between present 50 elements solution
and those by finite strip method [45].

results by Craig and Dawe. However, no Raleigh Ritz solution was given in [45] for comparison.

Since all the present analyses were carried out using consistent mass matrices, it is of interest to see the results obtained using lumped mass matrix for the same anisotropic case as shown in Table 4.10. The results show a discrepancy of about 6.9% for the thickness-to-length ratio (h/a) of 0.1 between the present element and the finite strip results by Craig and Dawe [45].

It is noted that a relatively fine mesh of 72 elements were used and convergence of lumped mass results has not quite yet been achieved. It is expected with further refinement, the results between the consistent and lumped mass formulation will agree.

2.8. Anisotropic Square Plates

An anisotropic square plate with various fiber angles and with simply-supported as well as clamped boundary conditions were analyzed. All the shear coefficients (α_1, α_2), the material properties, and the definition of nondimensional frequency were assumed to be the same as the previous example (4.2.7).

The results for the first four mode nondimensional frequencies Ω for these plates were obtained using two meshes (4x4, and 5x5) of the present element for two thickness-to-length ratios ($h/a = 0.01$ and 0.1) at seven fiber angles ($\theta = 0, 15, 30, 45, 60, 75, 90$) for the clamped boundary condition in Table 4.11 and for the simply-supported boundary condition in Table 4.12.

Table 4.10 Nondimensional frequency Ω for an anisotropic (30 degrees) clamped plate [lumped mass].

h/a	Mode Number	16 el element	32 el element	50 el element	72 el element	%diff- erence*	Finite Strip [45]
0.01	1	19.896	20.846	21.087	21.171	0.23	21.22
	2	26.734	31.029	32.263	32.670	1.00	33.00
	3	36.482	42.237	47.308	48.999	2.84	50.43
	4	36.531	45.898	49.069	50.253	1.87	51.21
0.10	1	13.809	14.653	14.989	15.263	-6.88	14.28
	2	18.251	21.137	22.424	23.148	-4.08	22.24
	3	20.546	24.252	26.099	27.542	-3.38	26.64
	4	21.969	26.083	29.014	30.566	1.43	31.01

* percentage difference between present 72 elements solution and those by finite strip method [45].

Table 4.1. First four mode natural frequencies Ω for various fiber angles h/a and thickness-to length ratios (h/a) for a clamped square anisotropic plate.

h/a	Modes												
	1			2			3			4			
	elements 32	elements 50	%diff- erence*	elements 32	elements 50	%diff- erence*	elements 32	elements 50	%diff- erence*	elements 32	elements 50	%diff- erence*	
0.01	0	23.84	23.83	0.02	31.09	31.04	0.17	46.57	46.28	0.62	61.88	61.83	0.08
	15	22.98	22.97	0.01	31.42	31.40	0.09	47.55	47.45	0.23	58.70	58.66	0.07
	30	23.26	23.25	0.01	33.07	33.02	0.14	50.52	50.39	0.27	51.36	51.30	0.13
	45	20.43	20.41	0.09	34.88	34.83	0.16	46.63	46.52	0.24	51.92	51.76	0.31
	60	21.26	21.25	0.04	33.07	33.02	0.14	50.52	50.39	0.27	51.36	51.30	0.13
	75	22.98	22.97	0.03	31.42	31.40	0.09	47.55	47.45	0.23	58.70	58.66	0.07
90	23.84	23.83	0.02	31.09	31.04	0.17	46.57	46.28	0.62	61.88	61.83	0.08	
0.1	0	16.31	16.31	0.01	23.27	23.24	0.11	30.66	30.65	0.04	33.48	33.36	0.36
	15	16.04	16.04	0.01	23.44	23.43	0.05	30.27	30.27	0.02	33.57	33.52	0.15
	30	15.45	15.44	0.03	24.20	24.18	0.07	29.17	29.16	0.04	33.54	33.49	0.13
	45	15.13	15.13	0.04	24.99	24.97	0.08	28.21	28.18	0.08	33.35	33.30	0.15
	60	15.45	15.44	0.03	24.20	24.18	0.07	29.17	29.16	0.04	33.54	33.49	0.13
	75	16.04	16.04	0.01	23.44	23.43	0.05	30.27	30.27	0.02	33.57	33.52	0.15
90	16.31	16.31	0.01	23.27	23.24	0.11	30.66	30.65	0.04	33.48	33.36	0.36	

* percentage difference between 32 and 50 element results.

Table 4.12 First four mode natural frequencies Ω for various fiber angles θ and thickness-to length ratios (h/a) for a simply-supported square anisotropic plate.

h/a	θ	Modes											
		1			2			3			4		
		Ω elements		Σ diff- erence*	Ω elements		Σ diff- erence*	Ω elements		Σ diff- erence*	Ω elements		Σ diff- erence*
0.01	0	32	50		32	50		32	50		32	50	
	15	11.06	11.06	0.00	17.73	17.73	0.02	32.03	31.97	0.18	40.11	40.10	0.01
	30	11.25	11.24	0.06	19.22	19.20	0.09	33.75	33.71	0.13	38.43	38.42	0.03
	45	11.82	11.79	0.25	21.89	21.87	0.10	35.14	35.08	0.18	36.24	36.23	0.04
	60	12.19	12.15	0.37	22.93	22.93	0.02	33.78	33.65	0.36	37.17	37.15	0.05
	75	11.82	11.79	0.25	21.89	21.87	0.10	35.14	35.08	0.18	36.24	36.23	0.04
0.1	0	11.06	11.06	0.00	17.73	17.73	0.02	32.03	31.97	0.18	40.11	40.10	0.01
	15	9.85	9.85	0.00	15.63	15.63	0.02	26.33	26.33	0.00	26.39	26.36	0.13
	30	9.99	9.98	0.05	16.67	16.66	0.07	25.86	25.86	0.01	27.25	27.23	0.08
	45	10.40	10.38	0.19	18.42	18.40	0.07	24.80	24.78	0.09	28.10	28.09	0.03
	60	10.65	10.62	0.29	19.06	19.06	0.01	24.30	24.25	0.19	28.36	28.34	0.05
	75	10.40	10.38	0.19	18.42	18.40	0.07	24.80	24.78	0.09	28.10	28.09	0.03
0.1	0	9.85	9.85	0.00	15.63	15.63	0.02	26.33	26.33	0.00	26.39	26.36	0.13
	15	9.99	9.98	0.05	16.67	16.66	0.07	25.86	25.86	0.01	27.25	27.23	0.08
	30	10.40	10.38	0.19	18.42	18.40	0.07	24.80	24.78	0.09	28.10	28.09	0.03
	45	10.65	10.62	0.29	19.06	19.06	0.01	24.30	24.25	0.19	28.36	28.34	0.05
	60	10.40	10.38	0.19	18.42	18.40	0.07	24.80	24.78	0.09	28.10	28.09	0.03
	75	9.99	9.98	0.05	16.67	16.66	0.07	25.86	25.86	0.01	27.25	27.23	0.08

* Percentage difference between 32 and 50 element results.

SECTION V

CONCLUDING REMARKS

Three concluding remarks are drawn here.

First, a 12 d.o.f. symmetrically laminated beam finite element, the associated solution procedure, and a computer program have been developed for the stress and vibration analyses using microcomputers. The simplicity and efficiency of this development have been evaluated through a series of examples. The effect of shear deformation for a homogeneous and isotropic beam, the effect of shear deformation for an orthotropic (0/90/0) laminated beam, the effect of shear deformation for an anisotropic laminated beam have all been verified through comparison of results with alternative existing solutions. The distribution of transverse shearing stress through the thickness for a quasi-isotropic laminated beam has also been compared. The natural frequencies of an isotropic homogeneous plate and those of an anisotropic symmetrically laminated cantilever plate have been compared.

The program was written in Basic language and matrices were stored in the form of half-band for the static analysis. For the free vibration analysis, the Jacobi eigenproblem solver was used. To expedite the computation, the program can be compiled as object code. A hard copy of the plots can be obtained using the available "screen dumps." For static analysis, the program can be used to obtain numerical data and graphical plots of the distributions of deflection,

to validate the accuracy and efficiency of the present element. Furthermore, the comparisons served as an indicator of the range of applicability and limitations of the present element.

In addition to being used for the purpose of application to solve practical anisotropic plate problem with moderate thickness, the present formulation concept and procedure may be used as a model for transforming other displacement type of isotropic plate and shell finite elements to anisotropic elements with the inclusion of the effects of shear deformation and rotatory inertia.

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